Minimization Algorithms for Discrete Convex Functions

Akiyoshi Shioura
(Tohoku University)

Minimization of L-/M-convex Functions

- fundamental problems in discrete convex analysis
- many examples & applications
- various algorithmic approaches
  - Greedy, Scaling, Continuous Relaxation, etc.

Outline of Talk

- Overview of Discrete Convex Analysis
- Definitions of L-/M-convex Functions
- Algorithms for Unconstrained Minimization
  - Greedy
  - Scaling
  - Continuous Relaxation
- Algorithms for More Difficult Problems

Outline of Talk

- Overview of Discrete Convex Analysis
- Definitions of L-/M-convex Functions
- Algorithms for Unconstrained Minimization
  - Greedy
  - Scaling
  - Continuous Relaxation
- Algorithms for More Difficult Problems

Overview of Discrete Convex Analysis

Discrete Convex Analysis [Murota 1996]
--- theoretical framework for discrete optimization problems

discrete analogue of
Convex Analysis
in continuous optimization

generalization of Theory of
Matroid/Submodular Function
in discrete opitmization

- key concept: two discrete convexity: L-convexity & M-convexity
  - generalization of Submodular Set Function & Matroid
- various nice properties
  - local optimal ←→ global optimal
  - duality theorem, separation theorem, conjugacy relation
- set/function are discrete convex → problem is tractable

History of
Discrete Convex Analysis

1935: Matroid Whitney

1965: Polymatroid, Submodular Function Edmonds

1983: relation between Submodularity and Convexity
Lovász, Frank, Fujishige

1992: Valuated Matroid Dress, Wenzel

1996: Discrete Convex Analysis, L-/M-convexity Murota

1996-2000: variants of L-/M-convexity
Fujishige, Murota, Shioura

Applications

- Combinatorial Optimization
  - matching, min-cost flow, shortest path, min-cost tension
- Math economics / Game theory
  - allocation of indivisible goods, stable marriage
- Operations research
  - inventory system, queueing, resource allocation
- Discrete structures
  - finite metric space
- Algebra
  - polynomial matrix, tropical geometry

Outline of Talk

- Overview of Discrete Convex Analysis
- Definitions of L-/M-convex Functions
- Algorithms for Unconstrained Minimization
  - Greedy
  - Scaling
  - Continuous Relaxation
- Algorithms for More Difficult Problems

Definition of L♮-convex Fn

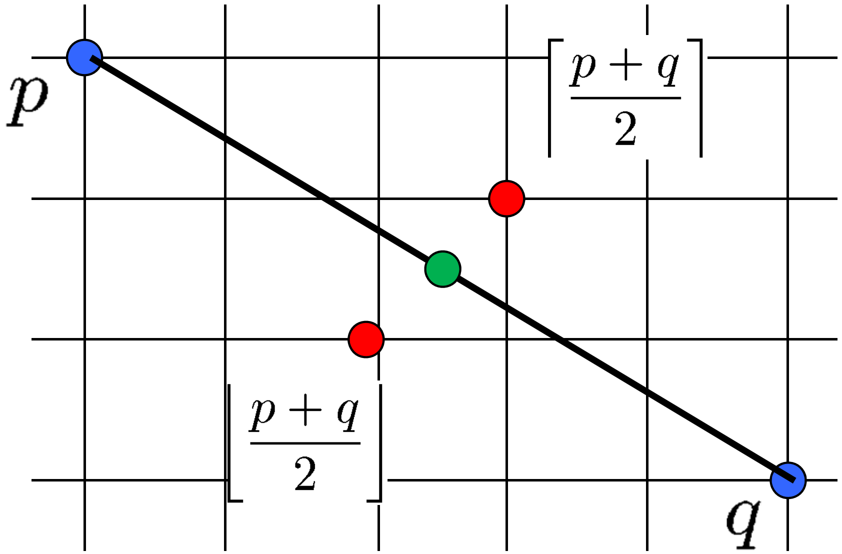
- L♮ -- L-natural, L=Lattice
- Def: $g: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is L♮-convex (Fujishige-Murota 2000)

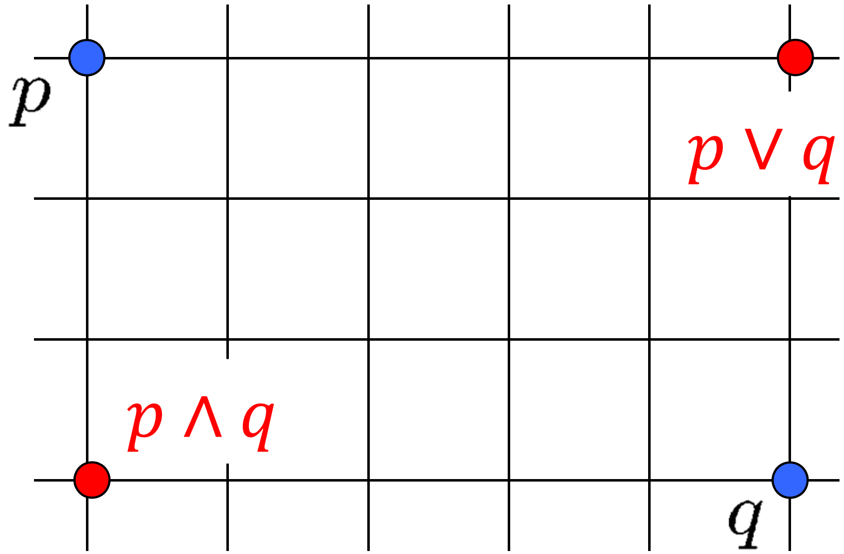
⟷ [discrete mid-point convexity]

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right) \quad (\forall p, q \in \mathbb{Z}^n)$$

⟷ integrally convex + submodular (Favati-Tardella 1990)

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \quad (\forall p, q \in \mathbb{Z}^n)$$





Examples of L♮-convex Fn

- univariate convex $\varphi: \mathbb{Z} \to \mathbb{R}$ ⇔ $\varphi(t-1) + \varphi(t+1) \geq 2\varphi(t)$
- separable-convex fn
- submodular set fn ⇔ L♮-conv fn with $\mathrm{dom}\, g = \{0,1\}^n$
- quadratic fn $g(p) = p^T A p$ is L♮-convex

  ⇔ $a_{ij} \leq 0 \ (i \neq j)$, $\sum_j a_{ij} \geq 0$

$$\begin{bmatrix} 4 & & & -1 \\ & 3 & -2 & \\ & -2 & 3 & -1 \\ -1 & & -1 & 5 \end{bmatrix}$$

- Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$
- min-cost tension problem

$$g(p) = \sum_{i=1}^{n} \varphi_i(p_i) + \sum_{i,j} \psi_{ij}(p_i - p_j)$$

(φ_i, ψ_{ij}: univariate conv fn)

Definition of M♮-convex Function

M♮-convex fn: a variant of M-convex fn

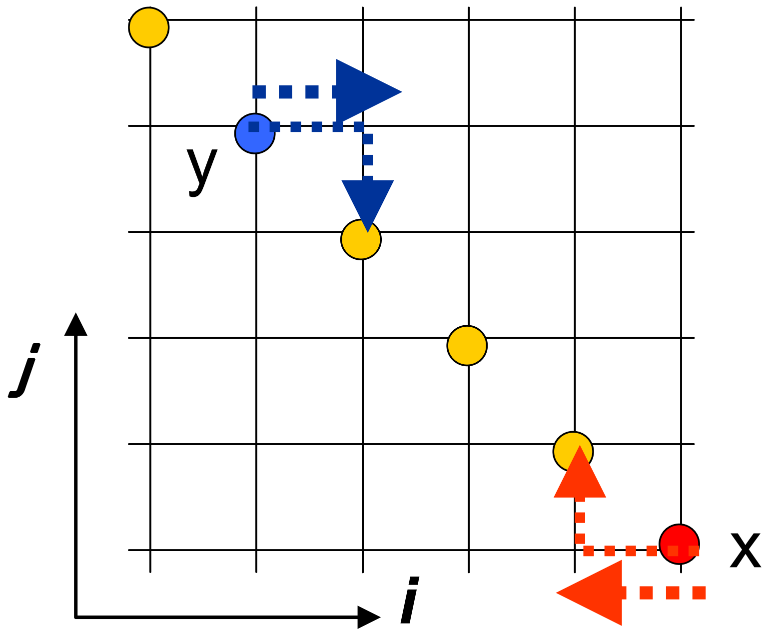
Def: $f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is M♮-convex ⟺

$\forall x, y \in \mathbb{Z}^n, \forall i: x(i) > y(i)$:

(i) $f(x) + f(y) \geq f(x - \chi_i) + f(y + \chi_i)$, or

(ii) $\exists j: x(j) < y(j)$ s.t. $f(x) + f(y) \geq f\left(x - \chi_i + \chi_j\right) + f(y + \chi_i - \chi_j)$

(Murota-Shioura99)



Examples of M$^\natural$-convex Functions

- Univariate convex $\varphi: \mathbb{Z} \to \mathbb{R}$ ⇔ $\varphi(t-1) + \varphi(t+1) \geq 2\varphi(t)$
- Separable convex fn on polymatroid:

  For integral polymatroid $P \subseteq \mathbb{Z}_+^n$ and univariate convex φ_i

  $$f(x) = \sum_{i=1}^{n} \varphi_i(x(i)) \quad \text{if} \quad x \in P$$

- Matroid rank function [Fujishige05]

  $f(X) = \max\{|Y| \mid Y: \text{independent set}, Y \subseteq X\}$ is M$^\natural$-concave

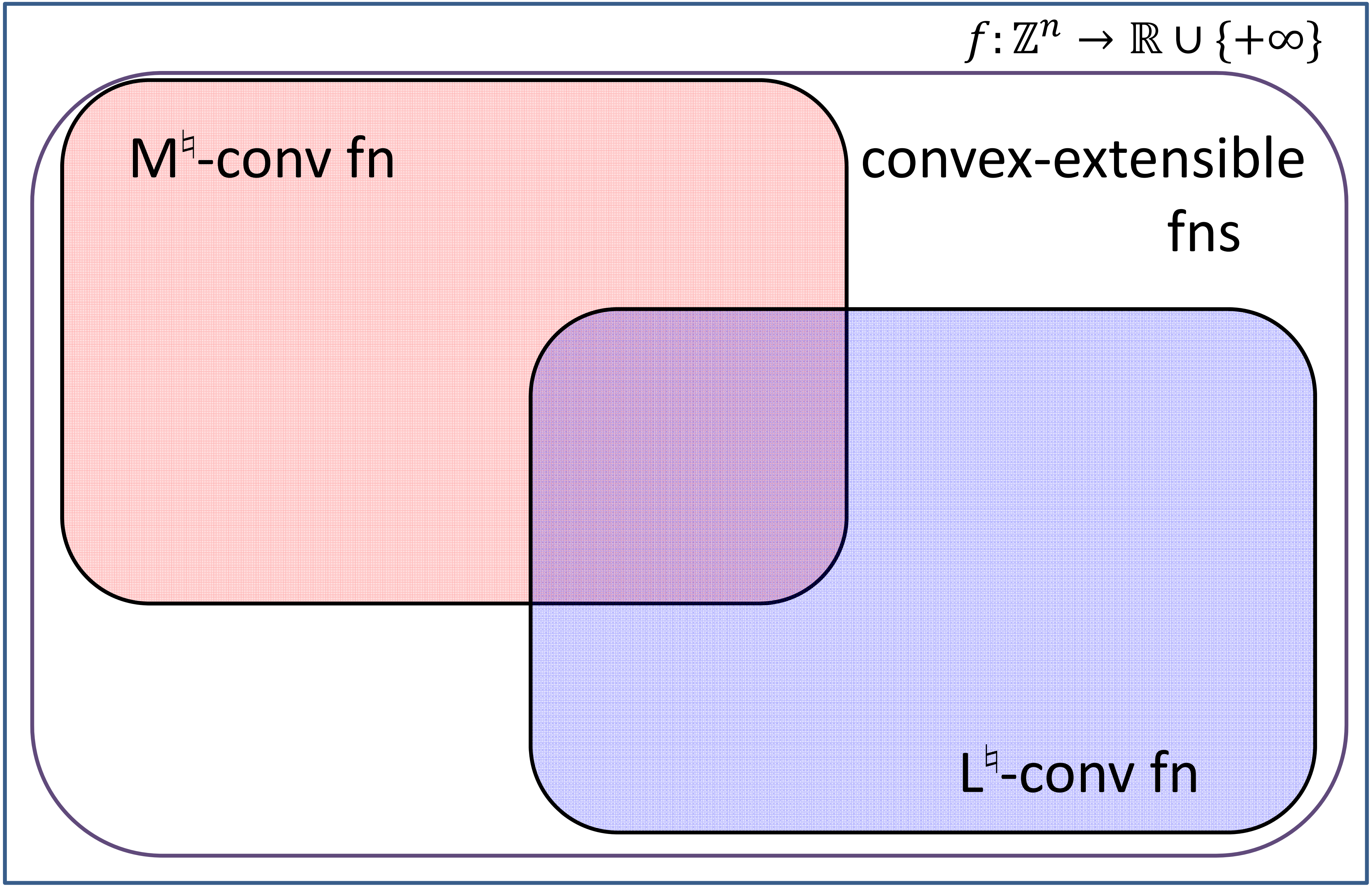
- Weighted rank function [Shioura09] $(w \geq 0)$

  $f(X) = \max\{w(Y) \mid Y: \text{independent set}, Y \subseteq X\}$ is M$^\natural$-concave

- Gross substitutes utility in math economics/game theory

  ⇔ M$^\natural$-concave fn on $\{0,1\}^n$ [Fujishige-Yang03]

Relationship of L-/M-convex Fns



Outline of Talk

- Overview of Discrete Convex Analysis
- Definitions of L-/M-convex Functions
- Algorithms for Unconstrained Minimization
  - Greedy
  - Scaling
  - Continuous Relaxation
- Algorithms for More Difficult Problems

Our Problems

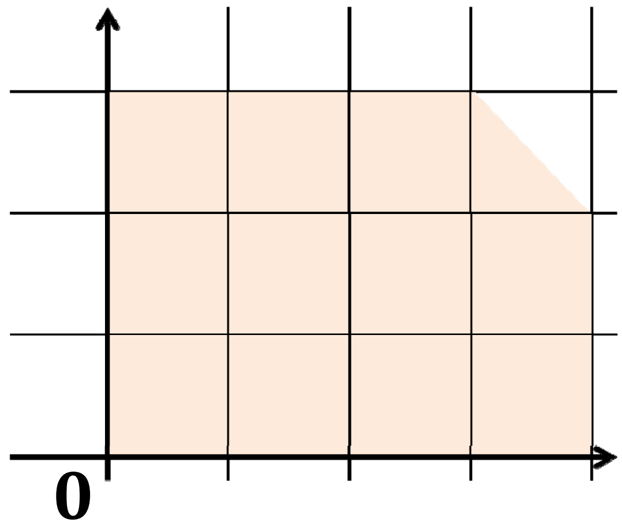
- Minimization of L$^\natural$-convex function

- Minimization of M$^\natural$-convex function
  - special case:

  $\mathbf{0} = (0, \ldots, 0)$ is unique minimal vector

  in dom f=$\{x | f(x) < +\infty\}$

  (←→ dom f is integral polymatroid)



Optimality Criterion for Minimization Problems

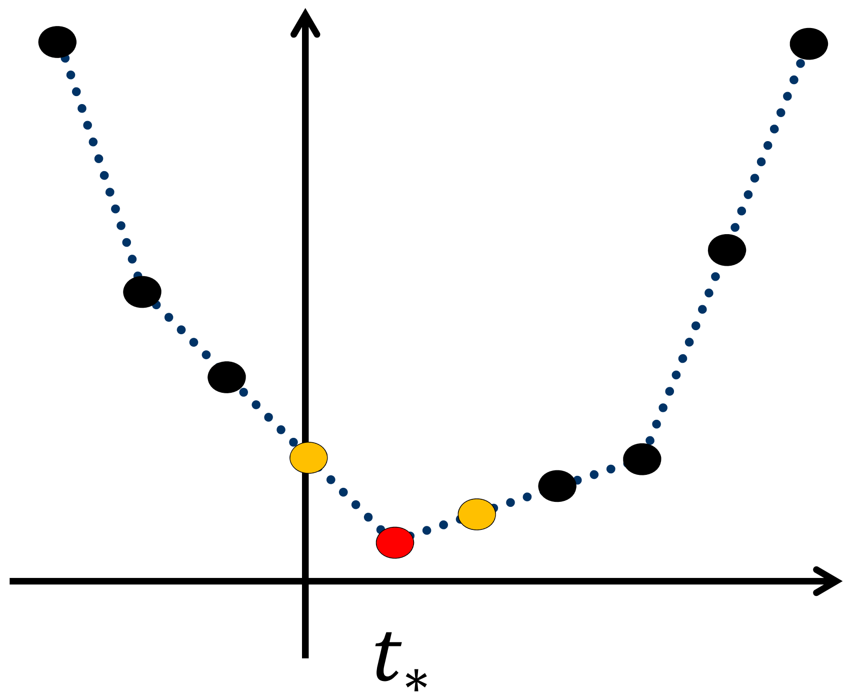
Optimality Criterion: General Case

Desirable property of "discrete convex" fn:

x: global opt ⇔ x: local opt w.r.t. some neighborhood $N(x)$

◆ univariate convex fn

Prop : t_*: global opt ⇔
local opt w.r.t $N(t_*) = t_* + \{0, \pm 1\}$



◆ n-variate "discrete convex" fn

- local opt → global opt?
  - NOT for convex-extensible fn
- which neighborhood?

Optimality Criterion: L♮-convex Function

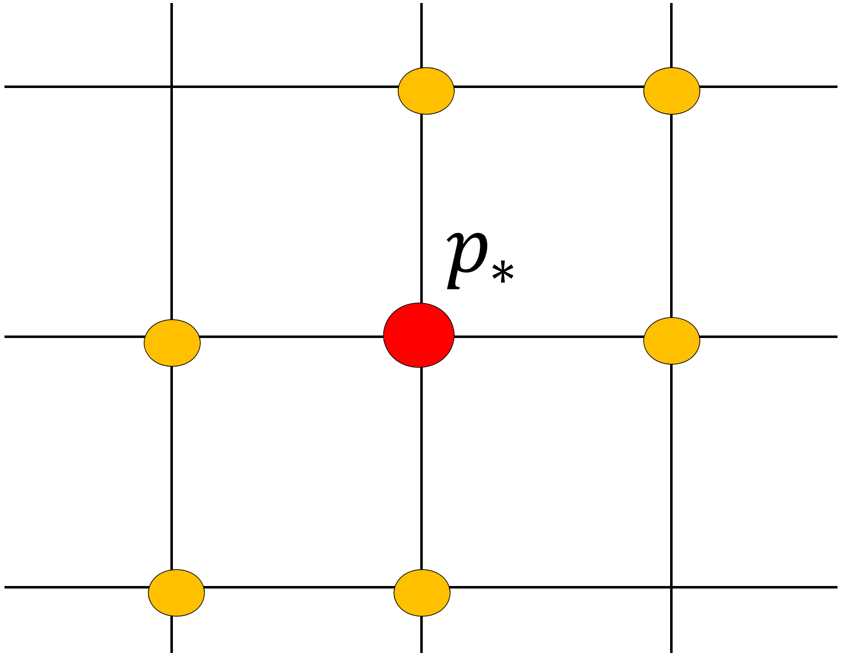
Thm :

p_*: global opt ←→ local opt in $p_* \pm q \;\; (q \in \{0,1\}^n)$

(Murota98, 03)

Local optimality check:

- need to check $O(2^n)$ vectors? --- No!
- can be reduced to
  submoduar set fn min --- poly time
  - $\rho_\pm(Y) = g(p_* \pm \chi_Y)$ is submodular set fn
  - p_* is local opt ←→ $\rho_\pm(Y)$ takes min at $Y = \emptyset$



Optimality Criterion: M$^\natural$-convex Function

Thm:

x_*: global opt

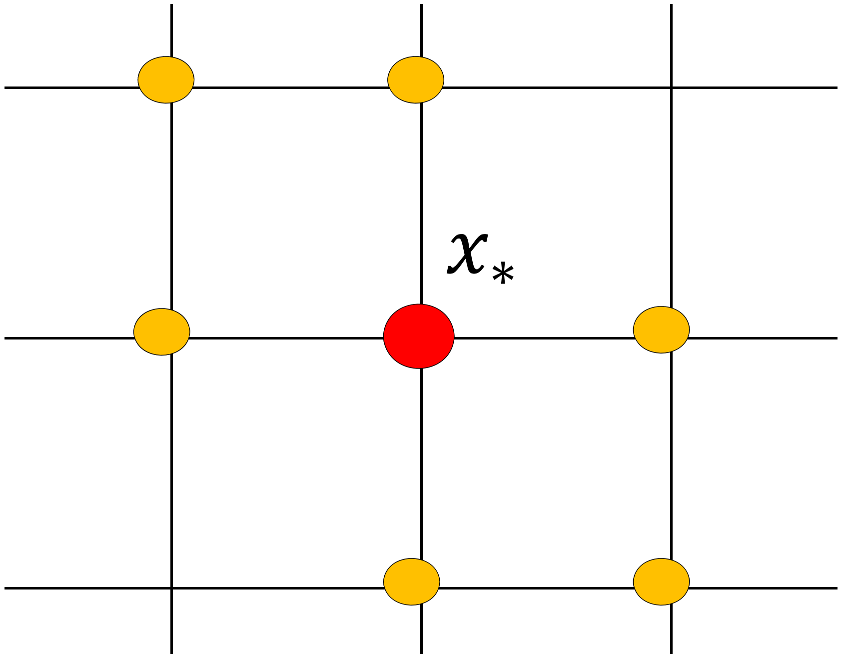
⇔ local opt in $x_* + \chi_i - \chi_j$ $(\forall i, j)$, $x_* \pm \chi_i$ $(\forall i)$

$x_* + (0, +1, 0, 0, -1, 0)$ $x_* + (0, \pm 1, 0, 0, 0, 0)$

(Murota96)

Local optimality check:

- $O(n^2)$ vectors → poly time



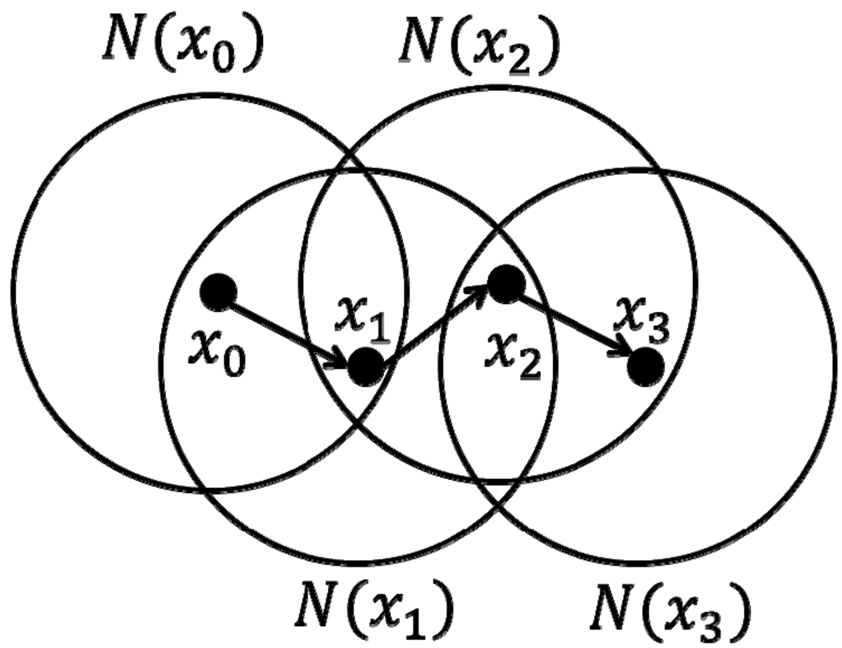
Greedy Algorithm

Greedy Algorithm: General Case

- Greedy Algorithm ≒ Steepest Descent Local Search
- "Global opt=Local opt" ➔ Greedy works

Repeat:
- find local min $y \in N(x)$
- set $x := y$

Stop if: x is local opt



- Greedy terminates in finite # of iters. (can be exponential)
- (pseudo)-poly. iteration?

Greedy Algorithm: L♮-convex Function

L♮-convex fn: global opt ←→ local opt w.r.t. $N(p) = p \pm \{0,1\}^n$

→ Greedy works with $N(p)$

Thm: p_0: initial sol., p_*: "nearest" global opt

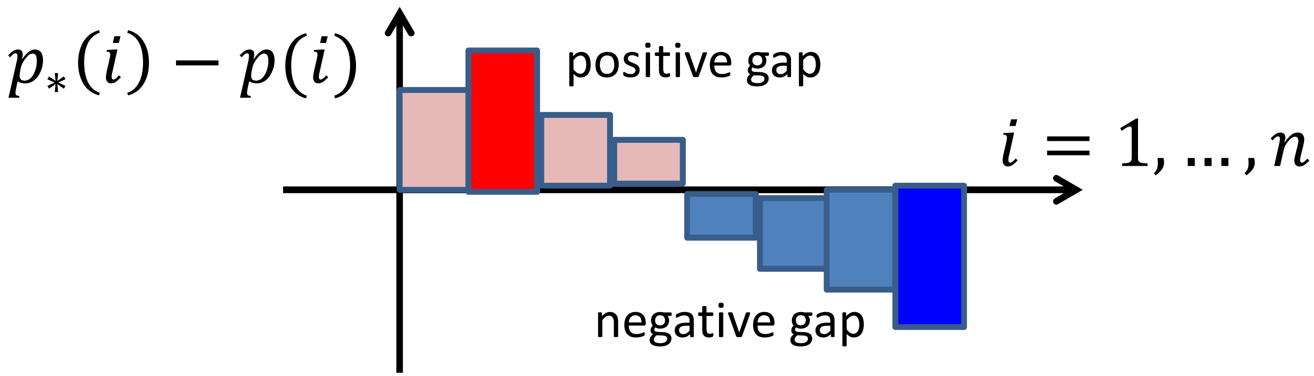
→ # of iter $\leq 2||p_* - p_0||_\infty$

(Kolmogorov-Shioura09)

Key Lemma: in each iteration,

"positive gap" $\max\{p_*(i) - p(i) \mid p_*(i) - p(i) > 0\}$ decreases, or

"negative gap" $\min\{p_*(i) - p(i) \mid p_*(i) - p(i) < 0\}$ increases



Greedy Algorithm: M♮-convex Function

M♮-convex fn:

global opt ←→ local opt w.r.t. $N(x) = x + \{\chi_i - \chi_j, +\chi_i, -\chi_j\}$

→ Greedy works with $N(x)$

Thm: x_0: initial sol., x_*: "nearest" global opt

→ # of iter $\leq ||x_* - x_0||_1$

(Murota03)

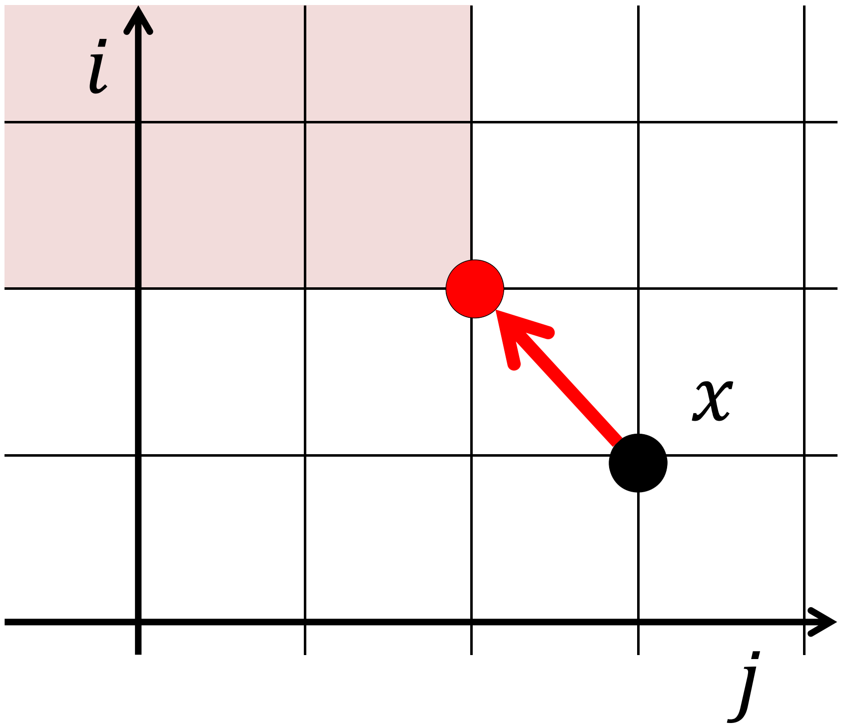
Minimizer Cut Thm:

$x + \chi_i - \chi_j \in N(x)$: local opt

→$\exists x_*$: global opt s.t.

$x_*(i) > x(i), \quad x_*(j) < x(j)$

(Shioura98)

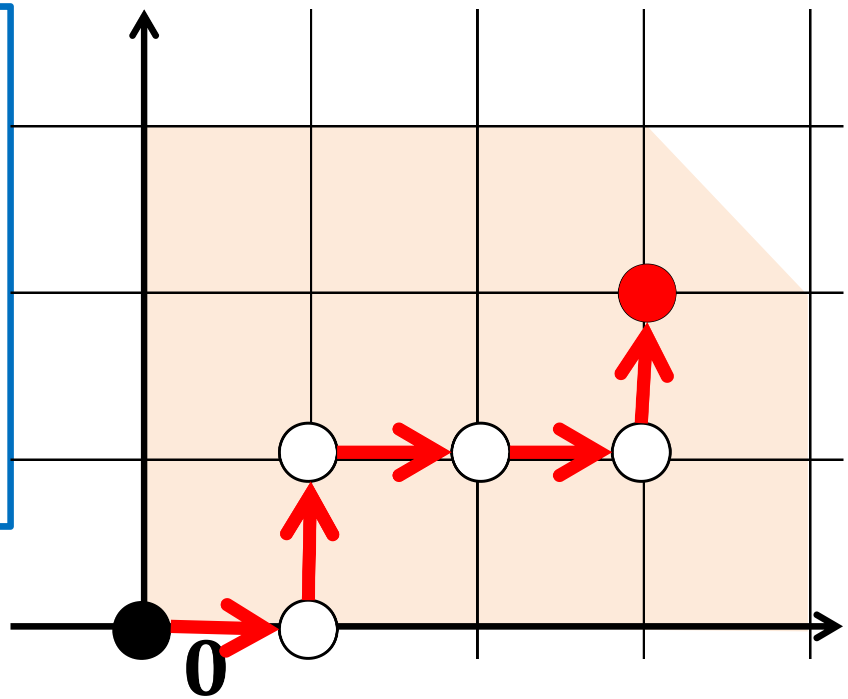


Greedy Algorithm for Special Case

Special Case: $\mathbf{0}$ is unique minimal in dom $f = \{x | f(x) < +\infty\}$

Initial vector: $y = (0, \ldots, 0)$
Repeat:
- find $i \in \arg\min\{f(y + \chi_i) | i \in N\}$
- set $y := y + \chi_i$

Stop if: $f(y + \chi_i) \geq f(y) \ (\forall i \in N)$



Minimizer Cut Thm 2:

(i) i minimizes $f(y + \chi_i)$ ➔ $\exists x_*$: opt. s.t. $x_*(i) > y(i)$

(ii) $f(y) \leq f(y + \chi_i)$ $(\forall i)$ ➔ $\exists x_*$: opt. s.t. $\sum_i x_*(i) \leq \sum_i y(i)$

Scaling and Proximity

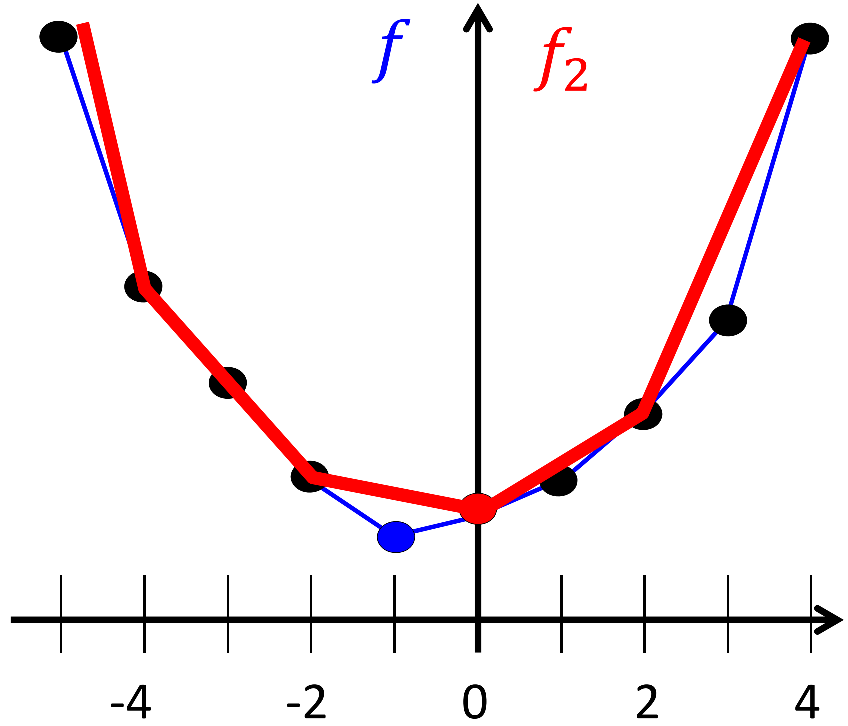
Scaling and Proximity: General Case

Scaling f_α of $f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

($\alpha \in \mathbb{Z}_+$: scaling parameter)

= restriction of f to $\alpha\mathbb{Z}^n$

$f_\alpha: \alpha\mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, $\quad f_\alpha(x) = f(x)$



"Proximity Thm":
global minimizer ● exists
in a neighborhood of
scaled (local) minimizer ●

➔ efficient algorithm

- univariate convex fn

Prop: |● — ●| ≦ α-1

- n-variate "discrete convex" fn

- ||● — ●|| is bounded? How large?

Scaling and Proximity: L♮-convex Function

Thm: $\forall p_\alpha$: scaled local minimizer, $\exists p_*$: global minimizer

s.t. $||p_* - p_\alpha||_\infty \leq (n-1)(\alpha-1)$

(Iwata-Shigeno03)

Prop: $\forall \alpha$: g_α is L♮-convex fn

➔ scaled (local) minimizer can be computed efficiently

➔ efficient scaling algorithm

Step 0: $\alpha :=$sufficiently large integer

Step 1: find minimizer x_α of g_α in a neighborhood of $x_{2\alpha}$

Step 2: if $\alpha = 1$, then stop (x is global opt)

Step 3: set $\alpha \coloneqq \alpha/2$; go to Step 1

Scaling and Proximity: M$^\natural$-convex Function

Thm: $\forall x_\alpha$: scaled local minimizer, $\exists x_*$: global minimizer
s.t. $||x_* - x_\alpha||_\infty \leq (n-1)(\alpha-1)$

(Moriguchi-Murota-Shioura02)

But: f_α is NOT M$^\natural$-convex

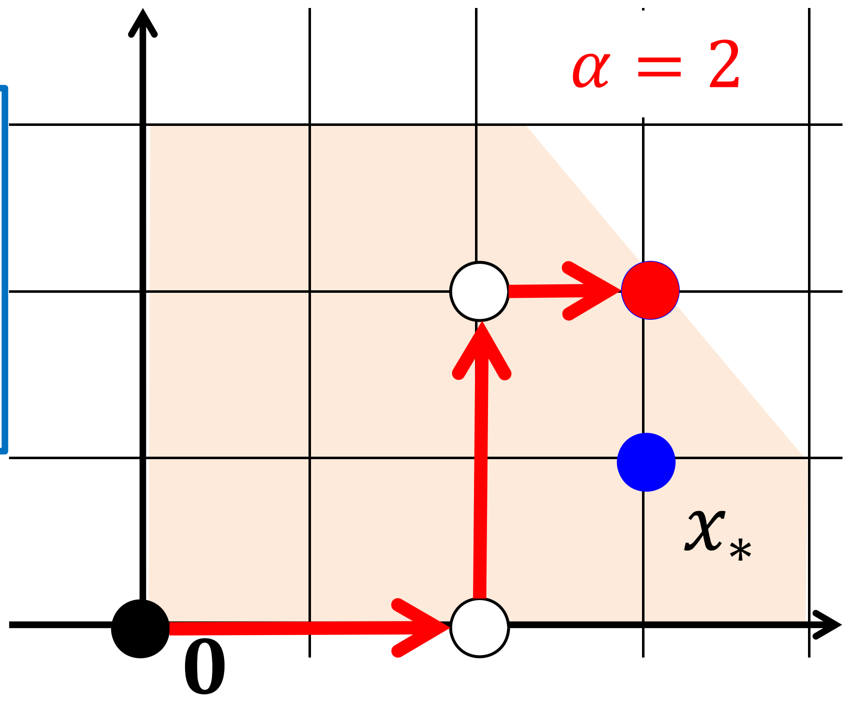
➔ difficult to compute a scaled local minimizer
➔simple scaling algo does not work
➔apply scaling approach in a different way

Scaling Algorithm for Special Case

Special Case: $(0, \ldots, 0)$ is unique minimal vector in dom f

apply scaling technique to Greedy Algo

Update of x using step size α:
- if $f(x + \alpha\chi_i) < \infty$, set $x := x + \alpha\chi_i$
- otherwise, set $x := x + \beta\chi_i$ with maximum β under $f(x + \beta\chi_i) < \infty$



Prop: x_α: output of scaled greedy algo,
$\exists x_*$: global minimizer
s.t. $||x_* - x_\alpha||_\infty \le \alpha - 1$

➔ efficient algorithm

※ can be extended to general M$^\natural$-convex fn

Continuous Relaxation and Proximity

Continuous Relaxation and Proximity: General Case

Assumption: convex fn $\tilde{f}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$

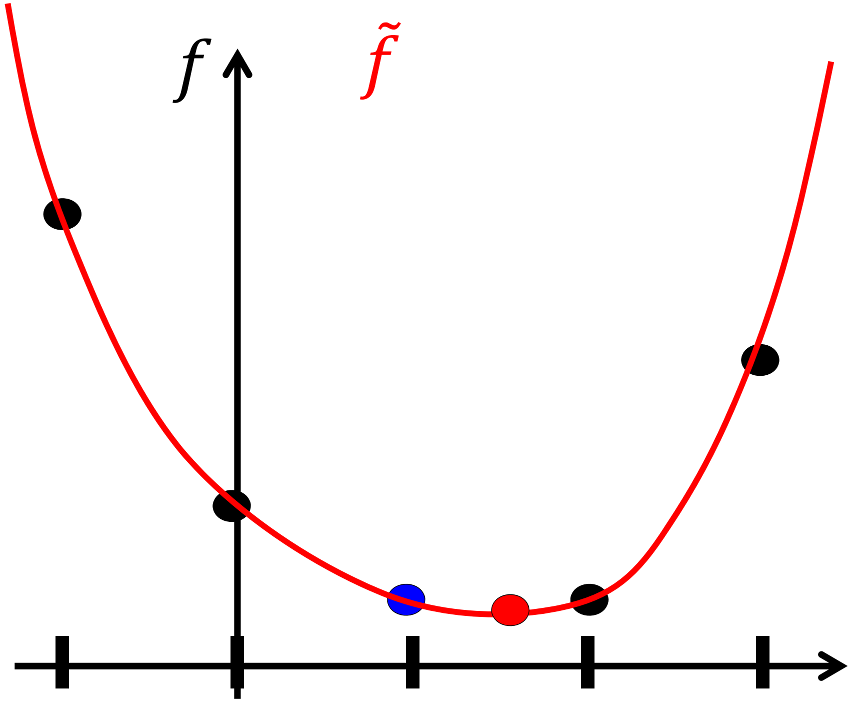
with $\tilde{f}(x) = f(x)$ $(\forall x \in \mathbb{Z}^n)$ is given

"Proximity Thm" :

int. minimizer ● exists

in a neighborhood of real minimizer ●

➔ efficient algorithm



◆ univariate convex fn

Prop : |●—●|< 1

◆ n-variate "discrete convex" fn

- ||●—●|| is bounded? How large?

Continuous L$^\natural$-convex Function

Assumption: continuous L$^\natural$-convex fn $\tilde{g}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\tilde{g}(p) = g(p)$ $(\forall p \in \mathbb{Z}^n)$ is given

Def: convex fn $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is continuous L$^\natural$-convex
⇔ $\hat{g}: \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ is submodular

$$\hat{g}(p_0, p_1, \ldots, p_n) = g(p_1 - p_0, \ldots, p_n - p_0)$$

(Murota-Shioura00,04)

Prop:

- restriction of cont. L$^\natural$-conv. fn on $\mathbb{Z}^n$ ➔ discrete L$^\natural$-conv.
- $\forall$ discrete L$^\natural$-conv. fn g, $\exists$ cont. L$^\natural$-conv. fn f s.t. $f(p) = g(p)$ $(\forall p \in \mathbb{Z}^n)$

Continuous Relaxation and Proximity: L♮-convex Function

Assumption: continuous L♮-convex fn $\tilde{g}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$
with $\tilde{g}(p) = g(p)$ $(\forall p \in \mathbb{Z}^n)$ is given

Thm: $\forall p_{\mathbb{R}}$: real minimizer, $\exists p_*$: integral minimizer
s.t. $||p_* - p_{\mathbb{R}}||_\infty \le n - 1$

(Moriguchi-Tsuchimura09)

if $p_{\mathbb{R}}$ can be computed efficiently (e.g., quadratic $\tilde{g}$)
➔ efficient algorithm for int. minimizer

Continuous M$^\natural$-convex Function

Assumption: continuous M$^\natural$-convex fn $\tilde{f}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$
with $\tilde{f}(x) = f(x)$ ($\forall x \in \mathbb{Z}^n$) is given

Def: convex fn $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is continuous M$^\natural$-convex $\Leftrightarrow$
$\forall x, y \in \mathbb{Z}^n, \forall i: x(i) > y(i), \exists \lambda_0 \in \mathbb{R}_+$:
(i) $f(x) + f(y) \geq f(x - \lambda \chi_i) + f(y + \lambda \chi_i)$ ($\forall \lambda \in [0, \lambda_0]$), or
(ii) $\exists j: x(j) < y(j)$ s.t.
$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j)$ ($\forall \lambda \in [0, \lambda_0]$)

(Murota-Shioura00,04)

Prop: $\forall$ discrete M$^\natural$-conv. fn g, $\exists$ cont. M$^\natural$-conv. fn f
s.t. $f(x) = g(x)$ ($\forall x \in \mathbb{Z}^n$)

※restriction of cont. M$^\natural$-conv. fn on $\mathbb{Z}^n$ is NOT discrete M$^\natural$-conv.

Continuous Relaxation and Proximity: M$^\natural$-convex Function

Assumption: continuous M$^\natural$-convex fn $\tilde{f}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$

with $\tilde{f}(x) = f(x)$ $(\forall x \in \mathbb{Z}^n)$ is given

Thm: $\forall x_{\mathbb{R}}$: real minimizer, $\exists x_*$: integral minimizer

s.t. $||x_* - x_{\mathbb{R}}||_\infty \le n - 1$

➔ $||x_* - x_{\mathbb{R}}||_1 \le n(n-1)$ (Moriguchi-Shioura-Tsuchimura11)

Special case: separable convex fn on polymatroid:

$$f(x) = \sum_{i=1}^{n} \varphi_i(x(i)) \quad \text{if} \quad x \in P$$

Thm: $\forall x_{\mathbb{R}}$: real minimizer, $\exists x_*$: integral minimizer

s.t. $||x_* - x_{\mathbb{R}}||_1 \le 2(n-1)$

if $x_{\mathbb{R}}$ can be computed efficiently (e.g., quadratic $\tilde{f}$)

➔ efficient algorithm for int. minimizer

Outline of Talk

- Overview of Discrete Convex Analysis
- Definitions of L-/M-convex Functions
- Algorithms for Unconstrained Minimization
  - Greedy
  - Scaling
  - Continuous Relaxation
- Algorithms for More Difficult Problems

Minimization of Sum of Two M$^\natural$-convex Fns

- Minimization of Sum of two M$^\natural$-convex fns $f_1, f_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$
  - sum of two M$^\natural$-convex fns is NOT M$^\natural$-convex
  - contains Polymatroid constrained problem:

Minimize $f_1(x)$ sub. to $x \in P$

→ Minimize $f_1(x) + f_2(x)$

where $f_2(x) = 0$ (if $x \in P$), $= +\infty$ (otherwise)

  - generalization of polymatroid intersection problem

- poly.-time solvable
  - polymatroid intersection algorithms can be extended
  - use new techniques & analysis

(Murota96,99,Iwata-Shigeno03,Iwata-Moriguchi-Murota05)

Minimization of Sum of Many M$^\natural$-convex Fns

- Minimization of Sum of more than two M$^\natural$-convex fns

$$f_1, \dots, f_m : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$$

  - contains Polymatroid constrained problem:

    Minimize $\sum_{j=1}^{m-1} f_j(x)$ sub. to $x \in P$

  - generalization of three polymatroid intersection problem
    - NP-hard

- (1-1/e)-approximation (for maximization version)

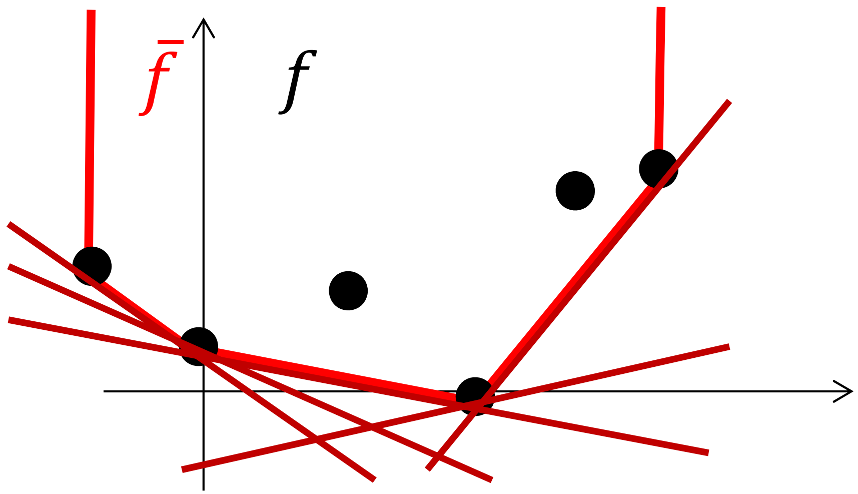
  for monotone $f_1, \dots, f_{m-1}$ & polymatroid const. (Shioura09)
  - continuous relaxation + pipage rounding (Calinescu et al. 07)
  - Key Property: convex closure of M$^\natural$-convex fn can be computed in poly-time ➔cont. relaxation in poly-time

Convex Closure of M♮-convex Fn

convex closure $\bar{f}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of $f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

--- point-wise maximal convex fn satisfying $\bar{f}(y) \le f(y)\ (\forall y \in \mathbb{Z}^n)$

$$\bar{f}(x) = \max\{p^T x + \alpha \mid p \in \mathbb{R}^n, \alpha \in \mathbb{R}, p^T y + \alpha \le f(y)(\forall y \in \mathbb{Z}^n)\}$$



Define $g(p) = \min\{f(y) - p^T y | y \in \mathbb{Z}^n\}$

➔ $\bar{f}(x) = \max\{p^T x + g(p) \mid p \in \mathbb{R}^n\}$

Prop: (i) restriction of g on $\mathbb{Z}^n$ is L♮-concave

(ii) if f is integer-valued

➔ $\max\{p^T x + g(p) \mid p \in \mathbb{R}^n\}$ has integral opt

➔ reduced to L♮-concave fn maximization

M$^\natural$-concave Function Maximization with Knapsack Constraints

- Maximization of M$^\natural$-concave fn $f: \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$
  under knapsack constraints $c_j^{\mathrm{T}} x \leq b_j \ (j = 1, \ldots, m)$
  - NP-hard
- polynomial-time approximation scheme (Shioura11)
  - continuous relaxation + simple rounding
  - near integrality of continuous opt. solution
  - Key Property: convex closure of M$^\natural$-convex fn can be computed in poly-time ➔cont. relaxation in poly-time