Introduction to Discrete Convex Analysis

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Discrete Convex Analysis

Discrete Convex Analysis [Murota 1996]
--- theoretical framework for discrete optimization problems

discrete analogue of Convex Analysis in continuous optimization

generalization of Theory of Matroid/Submodular Function in discrete optimization

- key concept: two discrete convexity: L-convexity & M-convexity
  - generalization of Submodular Set Function & Matroid
- various nice properties
  - local optimal $\leftrightarrow$ global optimal
  - duality theorem, separation theorem, conjugacy relation
- set/function are discrete convex $\Rightarrow$ problem is tractable
Applications

- **Combinatorial Optimization**
  - matching, min-cost flow, shortest path, min-cost tension
- **Math economics / Game theory**
  - allocation of indivisible goods, stable marriage
- **Operations research**
  - inventory system, queueing, resource allocation
- **Discrete structures**
  - finite metric space
- **Algebra**
  - polynomial matrix, tropical geometry
History of Discrete Convex Analysis

1935: Matroid
1965: Polymatroid, Submodular Function
1983: Submodularity and Convexity
1992: Valuated Matroid
1996: Discrete Convex Analysis, L-/M-convexity
1996-2000: variants of L-/M-convexity

1971: discretely convex function
1990: integrally convex function
Today’s Talk

• fundamental properties of M-convex & L-convex functions
• comparison with other discrete convexity
  – convex-extensible fn
  – Miller’s discretely convex fn
  – Favati-Tardella’s integrally convex fn
Outline of Talk

• Overview of Discrete Convex Analysis
• Desirable Properties of Discrete Convexity
• convex-extensible fn
• Miller’s discretely convex fn
• Favati-Tardella’s integrally convex fn
• M-convex & L-convex fns
• duality and conjugacy theorems for discrete convex fn
Desirable Properties of Discrete Convexity
Important Properties of Convex Fn

• optimality condition by local property
  $x$: local minimum in some neighborhood $\Rightarrow$ global minimum

• conjugacy relationship
  – conjugate of convex fn $\Rightarrow$ convex fn

• duality theorems
  – Fenchel duality
  – separation theorem
Desirable Properties of Discrete Convex Fn

- discrete convexity = “convexity” for functions $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$
  - convex extensibility
    - can be extended to convex fn on $\mathbb{R}^n$
  - optimality condition by local property
    - local minimum $\Rightarrow$ global minimum
      - local minimality depends on choice of neighborhood
  - duality theorems
    - “discrete” Fenchel duality
    - “discrete” separation theorem
  - conjugacy relationship
    - conjugate of “discrete” convex fn $\Rightarrow$ “discrete” convex fn
Classes of Discrete Convex Fns

• convex-extensible fn
• discretely convex fn (Miller 1971)
• integrally convex fn (Favati-Tardella 1990)
• M-convex fn, L-convex fn (Murota 1995, 1996)

satisfy desirable properties?
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Convex-Extensible Function
Definition of Convex-Extensible Fn

- A natural candidate for “discrete convexity”
- Def: \( f: \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is convex-extensible
  \[ \iff \exists \tilde{f}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \text{ convex fn s.t. } \tilde{f}(x) = f(x) \ (\forall x \in \mathbb{Z}^n) \]
Definition of Convex-Extensible Set

- Def: $S \subseteq \mathbb{Z}^n$ is convex-extensible if
  - indicator fn $\delta_S: \mathbb{Z}^n \rightarrow \{0, +\infty\}$ is convex-extensible,
  - $\text{conv}(S) \cap \mathbb{Z}^n = S$ ("no-hole" condition)

![Convex-extensible examples](image)

- Convex-extensible
- NOT convex-extensible
Properties of Convex-Extensible Fn

• if $n=1$, satisfies various nice properties
  – convex-extensible $\iff f(x - 1) + f(x + 1) \geq 2f(x)$
  – local min=global min, conjugacy, duality, etc.
    $\Rightarrow$ desirable concept as discrete convexity

• if $n \geq 2$,
  – convex-extensible (by definition)
  – what else?
Bad Results of Conv.-Extensible Fn

- any function $f$ with $\text{dom } f = \{0,1\}^n$ is convex-extensible
  $\rightarrow$ no good structure
- local opt $\neq$ global opt: $\forall k \in \mathbb{Z}_+, \exists f$: convex-extensible fn s.t.
  $x$: local min in $\{z \in \mathbb{Z}^n \mid \|z - x\|_{\infty} \leq k\}$ but NOT global min

Example: $\text{dom } f = \mathbb{Z}_+^2$, $f(x_1, x_2) = \max\{x_1 - 3x_2, -2x_1 + 3x_2\}$

$x = (0,0)$: local min in $\{z \in \mathbb{Z}^n \mid \|z - x\|_{\infty} \leq 1\}$, $f(0,0) > f(2,1)$
Separable-Convex Function

- **Def:** \( f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) is separable-convex \( \iff \)

\[
f(x) = \sum_{i=1}^{n} \varphi_i(x(i)), \text{ each } \varphi_i : \mathbb{Z} \rightarrow \mathbb{R} \cup \{+\infty\} \text{ is discrete convex}
\]

- examples: \( \sum_{i=1}^{n} x(i)^2, -\sum_{i=1}^{n} \log x(i) \), etc.
- satisfy various nice properties
  - convex-extensible
  - local min w.r.t. \( \{ z \mid \| z - x \|_1 \leq 1 \} = \text{global min} \)
- but, function class is too small
  - e.g., \( \text{dom } f \) is integer interval
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Miller’s Discretely Convex Fn
Definition of Discretely Convex Fn

- defined by discretized version of convex inequality

| Def: | $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is discretely convex (Miller 1971) |

\[
\Leftrightarrow \forall x, y \in \mathbb{Z}^n, \alpha \in [0,1], \quad s \equiv \alpha x + (1 - \alpha)y \\
f(s) \leq \alpha f(x) + (1 - \alpha)f(y) \\
\min\{f(z) | z(i) = \lfloor s(i) \rfloor \text{ or } \lceil s(i) \rceil \quad \forall i \}
\]

| Prop: | $s \in \mathbb{Z}^n \Rightarrow f(s) \leq \alpha f(x) + (1 - \alpha)f(y)$ |
Definition of Discretely Convex Set

- Def: $S \subseteq \mathbb{Z}^n$ is discretely convex

$\iff$ indicator fn $\delta_S: \mathbb{Z}^n \rightarrow \{0, +\infty\}$ is discretely convex

$\iff \forall x, y \in S, \, \alpha \in [0,1], \, s \equiv \alpha x + (1 - \alpha)y$

$\exists z \in S \text{ s.t. } z(i) = \lfloor s(i) \rfloor \text{ or } \lceil s(i) \rceil \, (\forall i)$

discretely convex

NOT discretely convex
Property of Discretely Convex Fn

- **Thm:** \([\text{local min} = \text{global min}]\)
  
  \[x \in \arg \min \{ f(z) \mid \|z - x\|_\infty \leq 1 \} \]

  \[\iff x \in \arg \min \{ f(z) \mid z \in \mathbb{Z}^n \} \]

- **validity of descent alg for minimization**

  repeat: (i) find \(z \in N_\infty(x)\) with \(f(z) < f(x)\)

  (ii) update \(x := z\)

- **size of neighborhood** \(\{z \mid \|z - x\|_\infty \leq 1\}\) is \(3^n \text{ --- exponential}\)
Bad Result of Discretely Convex Fn

- **Fact:** discretely conv fn is NOT convex-extensible
  discretely conv set is NOT convex-extensible
  (not satisfy “no-hole” condition)

- **Example:** $S = \{ x \in \mathbb{Z}^3 \, | \, x_1 + x_2 + x_3 \leq 2, x_i \geq 0 (i = 1,2,3) \}$
  $\cup \{ (1,2,0), (0,1,2), (2,0,1) \}$

$\Rightarrow S$ is discretely convex, but has a “hole”

$\{(1,2,0) + (0,1,2) + (2,0,1)\} / 3 = (1,1,1) \notin S$
$f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$

convex-extensible fn

separable-conv. fn

discretely convex fn
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• **Favati-Tardella’s integrally convex fn**
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• duality and conjugacy theorems for discrete convex fn
Integrally Convex Function
Convex Closure of Discrete Fn

- **Def:** convex closure $\bar{f}: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ of $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$
  --- point-wise maximal convex fn satisfying $\bar{f}(y) \leq f(y) \ (\forall y \in \mathbb{Z}^n)$

\[
\bar{f}(x) = \min\{\sum_{y \in \text{dom } f} \alpha_y f(y) \mid \alpha_y \geq 0 \ (y \in \text{dom } f), \sum_y \alpha_y = 1, \sum_y \alpha_y y = x\}
\]

- convex closure is convex fn
Local Convex Closure of Discrete Fn

- Def: local convex closure $\tilde{f}: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ of $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$

$\iff$ collection of convex closure on each hypercube $\tilde{f}(x) = \min\{\sum_{y \in \text{HC}(x)} \alpha_y f(y) \}$

\[ \alpha_y \geq 0 \; (y \in \text{HC}(x)), \sum_y \alpha_y = 1, \sum_y \alpha_y y = x \]

$\text{HC}(x) = \{ y \in \mathbb{Z}^n \mid y(i) = \lfloor x(i) \rfloor \text{or} \lceil x(i) \rceil \; (\forall i) \}$

- $\tilde{f}(x) = f(x) \; (\forall x \in \mathbb{Z}^n)$

- local convex closure $\tilde{f}$ is not convex
Definition of Integrally Convex Fn

Def: $f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is integrally convex (Favati-Tardella 1990)

$\iff$ local conv. closure $\tilde{f}$ is convex fn $\iff \tilde{f} = \bar{f}$

- convex-extensible but NOT integrally convex
- convex-extensible & integrally convex
Properties of Integrally Convex Fn

• by definition, integrally convex fn is
  – convex-extensible
  – discretely convex
    \[ \text{⇒ local min w.r.t. } \{z \mid \|z - x\|_\infty \leq 1\} = \text{global min} \]
Bad Results of Integrally Convex Fn

• by definition, integrally convex fn is
  – convex-extensible
  – discretely convex

  ➔ local min w.r.t. \( \{z \mid \|z - x\|_{\infty} \leq 1\} \) = global min
  but, neighborhood contains \( 3^n \) vectors (exponential)

• any function \( f \) with \( \text{dom } f = \{0,1\}^n \) is integrally convex
  ➔ no good structure

• failure of “discrete” separation theorem
Separation Theorem for Convex Fn

- Separation Theorem:

  \[ f: \text{convex fn}, \ g: \text{concave fn}, \ f(x) \geq g(x) \ (\forall x \in \mathbb{R}^n) \]

  \[ \Rightarrow \exists \ \text{affine fn} \ p^T x + \alpha \ \text{s.t.} \ f(x) \geq p^T x + \alpha \geq g(x) \ (\forall x \in \mathbb{R}^n) \]

- Equivalent to Duality Theorem for nonlinear programming

  \[ \Rightarrow \ \text{efficient primal-dual-type algorithm} \]
Discrete Separation Thm for Discrete Convex Fn

• “Discrete” Separation Theorem:

\( f: \) “discrete convex” fn, \( g: \) “discrete concave” fn,

\[ f(x) \geq g(x) \quad (\forall x \in \mathbb{Z}^n) \]

\( \Rightarrow \exists \) affine fn \( ax + b \) s.t. \( f(x) \geq ax + b \geq g(x) \quad (\forall x \in \mathbb{Z}^n) \)

• equivalent to Duality Theorem for combinatorial optimization

\( \Rightarrow \) efficient primal-dual-type algorithm
Failure of Discrete Separation for Integrally Convex/Concave Fns

• \( \exists f: \text{ integrally convex, } g: \text{ integrally concave} \) s.t.
  \[
  f(x) \geq g(x) \quad (\forall x \in \mathbb{Z}^n)
  \]
  but \( \nexists \) affine fn \( p^T x + \alpha \) with
  \[
  f(x) \geq p^T x + \alpha \geq g(x) \quad (\forall x \in \mathbb{Z}^n)
  \]

\[
\begin{align*}
  f(x_1, x_2) &= \max\{0, x_1 + x_2 - 1\} \quad \text{--- integrally convex,} \nonumber \\
  g(x_1, x_2) &= \min\{x_1, x_2\} \quad \text{--- integrally concave,} \nonumber \\
  f(x_1, x_2) &\geq g(x_1, x_2) \quad (\forall (x_1, x_2) \in \mathbb{Z}^2), \text{ but } f(0.5, 0.5) < g(0.5, 0.5) \nonumber \\
  \Rightarrow \text{ no affine fn with } f(x) &\geq p^T x + \alpha \geq g(x) \quad (\forall x \in \mathbb{Z}^2)
\end{align*}
\]
$f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$

convex-extensible fn

integrally convex fn

separable-conv. fn

discretely convex fn
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L-convex Function
Definition of $\mathbb{L}^\|$-convex Fn

- $\mathbb{L}^\|$ -- L-natural, $L$=Lattice
- **Def**: $g : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is $\mathbb{L}^\|$-convex (Fujishige-Murota 2000)

\[\iff \text{[discrete mid-point convexity]}\]
\[
g(p) + g(q) \geq g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right) \quad (\forall p, q \in \mathbb{Z}^n)
\]

\[\iff \text{integrally convex + submodular (Favati-Tardella 1990)}\]
\[
g(p) + g(q) \geq g(p \lor q) + g(p \land q) \quad (\forall p, q \in \mathbb{Z}^n)
\]

※ $\mathbb{L}^\|$-convex $\Rightarrow$ int. convex $\Rightarrow$ conv.-extensible & discr. convex
Examples of $L^\mathbb{R}$-convex Fn

- univariate convex $\varphi: \mathbb{Z} \to \mathbb{R}$ \iff $\varphi(t - 1) + \varphi(t + 1) \geq 2\varphi(t)$
- separable-convex fn
- submodular set fn $\iff$ $L^\mathbb{R}$-conv fn with $\text{dom } g = \{0,1\}^n$

- quadratic fn $g(p) = p^T Ap$ is $L^\mathbb{R}$-convex $\iff a_{ij} \leq 0$ ($i \neq j$), $\sum_j a_{ij} \geq 0$

- Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

- min-cost tension problem

\[ g(p) = \sum_{i=1}^n \varphi_i(p_i) + \sum_{i,j} \psi_{ij}(p_i - p_j) \]

($\varphi_i, \psi_{ij}$: univariate discrete conv fn)
Optimality Condition by Local Property

• Thm: [local min = global min]

\[ g(p) \leq \min\{g(p + \chi_X), g(p - \chi_X)\} \quad (\forall X \subseteq \{1,2, \ldots, n\}) \]

\[ \iff g(p) \leq g(q) \quad (\forall q \in \mathbb{Z}^n) \]

\[ \chi_X(i) = \begin{cases} 1 & (i \in X) \\ 0 & (i \not\in X) \end{cases} \]

※ local minimality check can be done efficiently

\[ \rho(X) \equiv g(p + \chi_X), \mu(X) \equiv g(p - \chi_X) \]

⇒ \( \rho, \mu : \text{submodular set fns} \), minimization in poly-time
M-convex Function
Characterization of Convex Function

- Prop: [“equi-distant” convexity]

\[ f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \text{ is convex } \iff \forall x, y \in \mathbb{R}^n, \exists \delta > 0, \]

\[ f(x) + f(y) \geq f(x - \alpha(x - y)) + f(y + (\alpha(x - y))) \]

\[ (0 \leq \forall \alpha \leq \delta) \]
Definition of $\mathcal{M}$-convex Function

\[ f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\} \] is $\mathcal{M}$-convex \cite{Murota-Shioura99} \iff $\forall x, y \in \mathbb{Z}^n, \forall i: x(i) > y(i)$:

(i) $f(x) + f(y) \geq f(x - \chi_i) + f(y + \chi_i)$, \quad or \quad 
(ii) $\exists j: x(j) < y(j)$ s.t. $f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j)$

\[ \text{M=Matroid} \]
Examples of $\mathbb{M}^\|$-convex Functions

- Univariate convex $\varphi: \mathbb{Z} \to \mathbb{R}$ \iff $\varphi(t - 1) + \varphi(t + 1) \geq 2\varphi(t)$
- Separable convex fn on polymatroid:
  For integral polymatroid $P \subseteq \mathbb{Z}_+^n$ and univariate convex $\varphi_i$
  \[ f(x) = \sum_{i=1}^n \varphi_i(x(i)) \text{ if } x \in P \]
- Matroid rank function [Fujishige05]
  \[ f(X) = \max\{|Y| \mid Y: \text{independent set, } Y \subseteq X\} \text{ is } \mathbb{M}^\| \text{-concave} \]
- Weighted rank function [Shioura09] ($w \geq 0$)
  \[ f(X) = \max\{w(Y) \mid Y: \text{independent set, } Y \subseteq X\} \text{ is } \mathbb{M}^\| \text{-concave} \]
- Gross substitutes utility in math economics/game theory
  \iff $\mathbb{M}^\| \text{-concave fn on } \{0,1\}^n$ [Fujishige-Yang03]
Properties of $\mathbb{M}$-convex Fn

• Thm: [local min = global min]

\[
\begin{align*}
  f(x) & \leq f(x \pm \chi_j) \quad (\forall j), \\
  f(x) & \leq f(x + \chi_j - \chi_k) \quad (\forall j, k), \\
  \iff f(x) & \leq f(y) \quad (\forall y \in \mathbb{Z}^n)
\end{align*}
\]

※ size of neighborhood = $O(n^2)$

\[
\chi_j(i) = \begin{cases} 
  1 & (i = j) \\
  0 & (i \neq j)
\end{cases}
\]

• $\mathbb{M}$-convex $\Rightarrow$ int. convex $\Rightarrow$ conv.-extensible & discr. convex
convex-extensible fn

integrally convex fn

$M^-$-convex fn

separable -convex fn

$L^-$-convex fn

discretely convex fn

$f: \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$
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Conjugacy and Duality
Conjugacy for Convex Functions

• Legendre transformation for $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$:

$$f^\bullet(p) = \sup \{ \langle p, x \rangle - f(x) | x \in \mathbb{R}^n \} \quad (p \in \mathbb{R}^n)$$

convex fn is closed under Legendre transformation

• Thm:

$$f: \text{convex} \implies f^\bullet: \text{convex}, \quad (f^\bullet)^\bullet = f \text{ (if } f \text{ is closed})$$
Conjugacy for L-/M-Convex Functions

- integer-valued fn $f : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$
- discrete Legendre transformation:
  $$f^\bullet(p) = \sup\{\langle p, x \rangle - f(x) | x \in \mathbb{Z}^n \} \quad (p \in \mathbb{Z}^n)$$

L-convex fn and M-convex fn are conjugate

Thm:

(i) $f : \mathcal{M} \rightarrow \mathcal{L} \Rightarrow f^\bullet : \mathcal{L} \rightarrow \mathcal{M}, (f^\bullet)^\bullet = f$

(ii) $f : \mathcal{L} \rightarrow \mathcal{M} \Rightarrow f^\bullet : \mathcal{M} \rightarrow \mathcal{L}, (f^\bullet)^\bullet = f$

- generalization of relations in comb. opt.:
  - matroid $\leftrightarrow$ rank fn [Whitney 35]
  - polymatroid $\leftrightarrow$ submodular fn [Edmonds 70]
Fenchel Duality Theorem for Convex Fn

• Legendre transformation:
  \[ f^\bullet (p) = \sup \{ \langle p, x \rangle - f(x) | x \in \mathbb{R}^n \} \]
  \[ f^\circ (p) = \inf \{ \langle p, x \rangle - f(x) | x \in \mathbb{R}^n \} \]

• Fenchel Duality Thm:
  \[ f : \text{convex}, \ g : \text{concave} \implies \inf_{x \in \mathbb{R}^n} \{ f(x) - g(x) \} = \sup_{p \in \mathbb{R}^n} \{ g^\circ (p) - f^\bullet (p) \} \]
  \[ (f^\bullet : \text{convex}, \ g^\circ : \text{concave}) \]

• Fenchel duality thm \iff separation theorem
Fenchel-Type Duality Theorem for L-/M-Convex Fn

- discrete Legendre transformation:
  \[ f^\bullet(p) = \sup\{\langle p, x \rangle - f(x) | x \in \mathbb{Z}^n \} \]
  \[ f^\circ(p) = \inf\{\langle p, x \rangle - f(x) | x \in \mathbb{Z}^n \} \]

- Fenchel-type Duality Thm: [Murota 96,98]
  \[ f: M^\bullet\text{-convex}, \ g: M^\circ\text{-concave} \Rightarrow \]
  \[ \inf_{x \in \mathbb{Z}^n} \{f(x) - g(x)\} = \sup_{p \in \mathbb{Z}^n} \{g^\circ(p) - f^\bullet(p)\} \]
  \[(f^\bullet: L^\bullet\text{-convex}, \ g^\circ: L^\circ\text{-concave})\]
Discrete Separation Thm for \( L^\mathbb{Z} \) Convex Fn

- \( L^\mathbb{Z} \) Separation Theorem: [Murota 96,98]

\( f: L^\mathbb{Z} \)-convex fn, \( g: L^\mathbb{Z} \)-concave fn, \( f(p) \geq g(p) \) (\( \forall p \in \mathbb{Z}^n \))

\( \Rightarrow \exists \) affine fn s.t. \( f(p) \geq x^T p + \beta \geq g(p) \) (\( \forall p \in \mathbb{Z}^n \))
Discrete Separation Thm for $M$ \textbullet Convex Fn

- $M$ \textbullet Separation Theorem: [Murota 96,98]

  $f: M$ \textbullet-convex fn, $g: M$ \textbullet-concave fn, $f(x) \geq g(x)$ ($\forall x \in \mathbb{Z}^n$)

  $\Rightarrow$ $\exists$ affine fn s.t. $f(x) \geq p^T x + \alpha \geq g(x)$ ($\forall x \in \mathbb{Z}^n$)
Relation among Duality Thms

M♮ Separation Thm
\[ f(x) \geq p^T x + \alpha \geq g(x) \]

Fenchel-type Duality Thm
\[ \inf \{f(p) - g(p)\} = \sup \{g^\circ(x) - f^\bullet(x)\} \]

L♮ Separation Thm
\[ f^\bullet(p) \geq x^T p + \beta \geq g^\circ(p) \]

weight splitting thm for weighted matroid intersection [Iri-Tomizawa 76, Frank 81]

(poly)matroid intersection thm [Edmonds 70]

weighted matroid intersection thm [Iri-Tomizawa 76, Frank 81]

Fenchel-type duality thm for subm. fn [Fujishige 84]

discrete separation for subm. fn [Frank 82]