Quadratic M-convex and L-convex Functions

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Abstract

The concepts of L-convexity and M-convexity are introduced by Murota (1996) for functions defined over the integer lattice, and recently extended to polyhedral convex functions by Murota–Shioura (2000). L-convex and M-convex functions are deeply connected with well-solvability in combinatorial optimization problems with convex objective functions. In this paper, we consider these concepts for quadratic functions and the structure of the coefficient matrices of such quadratic functions. It is shown that quadratic L-convex and M-convex functions can be characterized by nice combinatorial properties of their coefficient matrices. The conjugacy relationship between quadratic L-convex and M-convex functions is also discussed.

Key words: discrete optimization, matroid, base polyhedron, quadratic function

1 Introduction

A quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ is defined by an $n \times n$ real symmetric matrix A as

$$f(x) = \frac{1}{2} x^{\mathrm{T}} A x \qquad (x \in \mathbf{R}^n). \tag{1.1}$$

It is well-known that f is convex if and only if A is positive semidefinite. In this paper, we consider quadratic convex functions equipped with "combinatorial structures." This is tantamount to investigating symmetric matrices with some combinatorial properties in addition to positive semidefiniteness.

Many different classes of matrices with combinatorial properties have been investigated in the area of matrix theory. A typical example is the class of

Preprint submitted to Elsevier ScienceJuly, 2001; Revised April, 2003; November, 2003

M-matrices that appear in various areas such as electric engineering, control engineering [26], numerical analysis [2,28], and mathematical economics [23]. An $n \times n$ real matrix A is called an M-matrix if $A = sI - B$, where B is an $n \times n$ nonnegative matrix and s is a real number greater than $\rho(B)$, the maximum modulus of an eigenvalue of B. In particular, a nonsingular M-matrix is a matrix such that each off-diagonal component is nonpositive and its inverse is a nonnegative matrix. An $n \times n$ diagonally-dominant symmetric M-matrix A can be characterized by the following property of a combinatorial nature:

$$a_{ij} \leq 0 \quad (\forall i, j \in N,\ i \neq j), \qquad \sum_{j=1}^{n} a_{ij} \geq 0 \quad (\forall i \in N), \tag{1.2}$$

where $N = \{1, 2, \ldots, n\}$. Note that a symmetric matrix A is called a compartmental matrix [1] if $-A$ satisfies the property (1.2). See [4] for mathematical properties of M-matrices.

The concept of Dirichlet forms is deeply connected with M-matrices. Indeed, a quadratic function $(1/2)x^{\mathrm{T}}Ax$ is a Dirichlet form (of finite dimension) if and only if A is a diagonally-dominant symmetric M-matrix. The Dirichlet form is a fundamental tool in probability theory (Markov process, in particular) and potential theory [5,10]. It is known that a Dirichlet form $f(x) = (1/2)x^{\mathrm{T}}Ax$ can be characterized by the following combinatorial property:

$$f(x) \geq f((\mathbf{0} \vee x) \wedge \mathbf{1}) \qquad (\forall x \in \mathbf{R}^n),$$

where $\mathbf{0}$ and $\mathbf{1}$ denote the vectors with all components being equal to zero and one, respectively, and for any $x, y \in \mathbf{R}^n$ the vectors $x \wedge y, x \vee y \in \mathbf{R}^n$ are given by

$$(x \wedge y)_i = \min\{x_i, y_i\}, \qquad (x \vee y)_i = \max\{x_i, y_i\} \qquad (i = 1, 2, \ldots, n).$$

Such a combinatorial property of Dirichlet forms can be seen as a reflection of the combinatorial structure of diagonally-dominant symmetric M-matrices. In more physical terms, the Dirichlet form is equivalent to (or represented as) electrical networks or to random walks. The present study will, hopefully, shed a new light on these fundamental concepts.

In the area of combinatorial optimization, on the other hand, "combinatorial convexity," i.e., convexity equipped with nice combinatorial properties has also been investigated (see, e.g., [6–8,15,16,25]). The relationship between submodular functions and convex functions was made clear through the works by Frank [7], Fujishige [8], and Lovász [15]. Miller [16] and Favati–Tardella [6] investigated classes of "discrete convex" functions defined over the integer lattice such that local optimality implies global optimality. In particular, problems with quadratic objective functions are often discussed in the literature [3,11–14]. Many of those quadratic problems have nice combinatorial structure and can be solved by efficient combinatorial algorithms. It is also hoped that the present study provides a common framework for such efficiently solvable quadratic combinatorial optimization problems.

The concepts of L-convexity and M-convexity are introduced for functions defined over the integer lattice by Murota [17–19], with a view to identifying nice combinatorial structure in combinatorial optimization problems with (general) nonlinear objective functions. Later, the concepts of $\mathrm{L}^\natural$-convexity and $\mathrm{M}^\natural$-convexity are introduced by Fujishige–Murota [9] and by Murota–Shioura [20], respectively, as variants of L-convexity and M-convexity. The concepts of L-convexity and M-convexity are extended to polyhedral convex functions defined over the real space by Murota–Shioura [21]. Variants of polyhedral L-convex and M-convex functions, called polyhedral $\mathrm{L}^\natural$-convex and $\mathrm{M}^\natural$-convex functions, are also introduced in [21].

Although the concepts of L-/$\mathrm{L}^\natural$-convexity and M-/$\mathrm{M}^\natural$-convexity are defined for polyhedral convex functions, we can naturally extend these concepts to quadratic functions. This extension provides a nice framework for well-solvable quadratic combinatorial optimization problems. The main aim of this paper is to investigate the combinatorial properties of quadratic L-/$\mathrm{L}^\natural$-convex and M-/$\mathrm{M}^\natural$-convex functions and the structure of their coefficient matrices. The results in this paper will be helpful in identifying well-solvable quadratic combinatorial optimization problems. Indeed, it can be shown by using our results that nonseparable quadratic objective functions treated in [11,13] are quadratic M-convex functions, which explains well-solvability of the quadratic combinatorial optimization problems in [11,13]. For combinatorial optimization problems with quadratic L-/M-convex objective functions, our results will be useful in developing efficient combinatorial algorithms.

We show that a quadratic function f given by (1.1) is $\mathrm{L}^\natural$-convex if and only if its coefficient matrix A is a diagonally-dominant symmetric M-matrix. A similar characterization of a quadratic L-convex function is given in terms of the sign pattern of the coefficient matrix. As for M-/$\mathrm{M}^\natural$-convexity, we show that a quadratic function f given by (1.1) with a nonsingular coefficient matrix A is $\mathrm{M}^\natural$-convex if and only if A satisfies the following combinatorial property:

$$x^{\mathrm{T}} a_i > \min\{0, \min_{j \neq i} x^{\mathrm{T}} a_j\} \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x)), \tag{1.3}$$

where $\mathrm{supp}^+(x) = \{i \in N \mid x_i > 0\}$ and a_i $(i \in N)$ denotes the i-th column vector of A. Moreover, this condition (1.3) turns out to be equivalent to the inverse of matrix A being a diagonally-dominant symmetric M-matrix.

This reveals a nice conjugacy relationship between quadratic L$^\natural$-convex functions and M$^\natural$-convex functions expressed in terms of the coefficient matrices, where the conjugacy means the Legendre–Fenchel conjugacy. The conjugacy relationship extends to nonquadratic convex functions [22].

We also consider L-/L$^\natural$-convex and M-/M$^\natural$-convex functions defined over $\mathbf{Z}^n$ (the integer lattice), and compare them to those over $\mathbf{R}^n$. This is to indicate a subtle difference between the discrete case and the continuous case, especially with M-convexity. It is shown that the classes of coefficient matrices of L-/L$^\natural$-convex functions over $\mathbf{Z}^n$ coincide with those for $\mathbf{R}^n$, whereas the coefficient matrices of quadratic M-/M$^\natural$-convex functions over $\mathbf{Z}^n$ constitute more restrictive classes than those for $\mathbf{R}^n$.

The organization of this paper is as follows. Definitions and examples of quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions are given in Section 2. Structures of quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions are investigated in Section 3, while the proofs are given in Section 6. The conjugacy relationship between quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions are discussed in Section 4. Characterizations of quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions over $\mathbf{Z}^n$ are provided in Section 5.

2 Definitions and Examples of Quadratic L-/M-convex Functions

We denote by $\mathbf{R}$ the set of reals, and by $\mathbf{Z}$ the set of integers. Throughout this paper, we assume that n is a positive integer, and denote $N = \{1, 2, \ldots, n\}$. For any finite set S, its cardinality is denoted by $|S|$. A family $\{N_1, N_2, \ldots, N_m\}$ of subsets of N is called a *subpartition* of N if $N_i \cap N_j = \emptyset$ for any distinct $i, j \in \{1, 2, \ldots, m\}$, and a *partition* if it is a subpartition and $\bigcup_{i=1}^m N_i = N$. The characteristic vector of a subset $S \subseteq N$ is denoted by χ_S ($\in \{0,1\}^n$), i.e., $(\chi_S)_i = 1$ for $i \in S$ and $(\chi_S)_i = 0$ for $i \in N \setminus S$. We denote $\chi_i = \chi_{\{i\}}$ for $i \in N$, in particular. We sometimes denote $\chi_0 = \mathbf{0}$. For $x \in \mathbf{R}^n$ and $S \subseteq N$, we define $x(S) = \sum_{i \in S} x_i$, and denote by $x[S] \in \mathbf{R}^S$ the restriction of x to S. The set of $n \times n$ real symmetric matrices is denoted by $\mathcal{S}_n$. For $A \in \mathcal{S}_n$ and $S \subseteq N$, the principal submatrix of A induced by the index set S is denoted by $A[S]$. Given a set of vectors $x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in \mathbf{R}^n$, we define

$$\mathrm{span}\{x^{(i)} \mid i = 1, 2, \ldots, m\} = \{\sum_{i=1}^m \alpha_i x^{(i)} \mid \alpha_i \in \mathbf{R}\ (i = 1, 2, \ldots, m)\}.$$

For a symmetric matrix $B \in \mathcal{S}_n$ and a linear subspace $K \subseteq \mathbf{R}^n$, we define a quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over K by

$$g(p) = \begin{cases} \frac{1}{2} p^{\mathrm{T}} B p & (p \in K), \\ +\infty & (p \notin K). \end{cases} \tag{2.1}$$

We call such a quadratic function *L-convex* if it satisfies the properties (LF1) and (LF2):

(LF1) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}\, g)$,
(LF2) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}\, g,\ \lambda \in \mathbf{R})$,

where $\mathrm{dom}\, g = \{p \in \mathbf{R}^n \mid g(p) < +\infty\}$. A quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by (2.1) is called *L$^\natural$-convex* if the function $\widehat{g} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$, where $\widehat{N} = \{0\} \cup N$, defined by

$$\widehat{g}(p_0, p) = g(p - p_0 \mathbf{1}) \qquad ((p_0, p) \in \mathbf{R}^{\widehat{N}})$$

is L-convex. We call a linear subspace $K \subseteq \mathbf{R}^n$ *L-convex* (resp., *L$^\natural$-convex*) if the indicator function $\delta_K : \mathbf{R}^n \to \{0, +\infty\}$ defined by

$$\delta_K(p) = \begin{cases} 0 & (p \in K), \\ +\infty & (p \notin K) \end{cases}$$

is L-convex (resp., L$^\natural$-convex).

Proposition 2.1 (cf. [21, Th. 3.23]) *A set $K \subseteq \mathbf{R}^n$ is an L-convex (resp., L$^\natural$-convex) linear subspace if and only if $K = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \ldots, m\}$ for some partition (resp., subpartition) $\{N_k \mid k = 1, 2, \ldots, m\}$ of N.*

For a symmetric matrix $A \in \mathcal{S}_n$ and a linear subspace $H \subseteq \mathbf{R}^n$, we define a quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over H by

$$f(x) = \begin{cases} \frac{1}{2} x^{\mathrm{T}} A x & (x \in H), \\ +\infty & (x \notin H). \end{cases} \tag{2.2}$$

We call such a quadratic function *M-convex* if it satisfies the property (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \quad (\forall \alpha \in (0, \alpha_0]),$$

where $\mathrm{supp}^+(z) = \{i \in N \mid z_i > 0\}$ and $\mathrm{supp}^-(z) = \{i \in N \mid z_i < 0\}$ for $z \in \mathbf{R}^n$. A quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by (2.2) is called M$^\natural$-*convex* if the function $\widehat{f} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$, where $\widehat{N} = \{0\} \cup N$, defined by

$$\widehat{f}(x_0, x) = \begin{cases} f(x) \ (x_0 = -x(N)), \\ +\infty \ \ (\text{otherwise}) \end{cases} \qquad ((x_0, x) \in \mathbf{R}^{\widehat{N}})$$

is M-convex. Also, we call a linear subspace *M-convex* (resp., *M$^\natural$-convex*) if its indicator function is M-convex (resp., M$^\natural$-convex).

Proposition 2.2 (cf. [21, Th. 3.3]) *A set $H \subseteq \mathbf{R}^n$ is an M-convex (resp., M$^\natural$-convex) linear subspace if and only if there exists a partition (resp., subpartition) $\{N_k \mid k = 1, 2, \ldots, m\}$ of N such that $H = \{x \in \mathbf{R}^n \mid x(N_k) = 0 \ (k = 1, 2, \ldots, m)\}$.*

Example 2.3 Quadratic M-/L-convex functions arise from electrical networks with linear resistors. Consider an electrical network with a branch set A and a node set V, and suppose that terminal set $T \subseteq V$, a current source vector $x \in \mathbf{R}^T$, and a potential vector $p \in \mathbf{R}^T$ are given. We denote the resistance of branch $a \in A$ by R_a, the potential at node $v \in V$ by p_v, the voltage across branch $a \in A$ by η_a, and the current in branch $a \in A$ by ξ_a. The current vector $\xi = (\xi_a \mid a \in A)$ satisfies Kirchhoff's current law (KCL) for x, which is represented as

$$\sum\{\xi_a \mid \text{branch } a \text{ leaves } v\} - \sum\{\xi_a \mid \text{branch } a \text{ enters } v\} = \begin{cases} x_v \ \ (v \in T), \\ 0 \ \ \ (v \in V \setminus T). \end{cases}$$

The voltage vector $\eta = (\eta_a \mid a \in A)$ satisfies Kirchhoff's voltage law (KVL) for p, expressed as the existence of a potential $\widetilde{p} \in \mathbf{R}^V$ such that $\widetilde{p}_v = p_v$ for $v \in T$ and $\eta_a = \widetilde{p}_u - \widetilde{p}_v$ for every branch $a = (u, v) \in A$. Then, the power (energy) consumed in the electrical network with current source $x \in \mathbf{R}^T$ is represented as

$$f(x) = \min\{\sum_{a \in A} \frac{1}{2} R_a \xi_a^2 \mid \xi \text{ satisfies KCL for } x\},$$

whereas the power consumed with potential $p \in \mathbf{R}^T$ is represented as

$$g(p) = \min\{\sum_{a \in A} \frac{1}{2R_a} \eta_a^2 \mid \eta \text{ satisfies KVL for } p\}.$$

It can be shown that f is quadratic M-convex and g is quadratic L-convex. Moreover, this construction with nonlinear resistors yields general (nonquadratic) M- and L-convex functions; see [22]. □

Example 2.4 In [11,13] it is shown that total weighted tardiness scheduling problems with high multiplicity can be reformulated as integer programs with quadratic objective functions, where such quadratic functions are given by coefficient matrices $A = (a_{ij})_{i,j \in N}$ of the form

$$a_{ij} = w_{\max\{i,j\}} \qquad (i, j \in N)$$

for some $w_1 \geq w_2 \geq \cdots \geq w_n$. This shows that the quadratic functions treated in [11,13] are M-convex functions over the integer lattice (see Section 5). □

3 Results

We describe the main theorems of this paper. The proofs are given in Section 6.

3.1 Quadratic L-convex/L$^\natural$-convex Functions

3.1.1 L-convex Functions

We first consider a special case where the function g is defined over the entire space $\mathbf{R}^n$:

$$g(p) = \frac{1}{2} p^{\mathrm{T}} B p \qquad (p \in \mathbf{R}^n). \tag{3.1}$$

The coefficient matrix of such a quadratic L-convex function has a simple sign pattern.

Theorem 3.1 *A quadratic function $g : \mathbf{R}^n \to \mathbf{R}$ given by* (3.1) *with $B \in \mathcal{S}_n$ is L-convex if and only if*

$$b_{ij} \leq 0 \quad (i, j \in N,\ i \neq j), \qquad \sum_{j=1}^{n} b_{ij} = 0 \quad (i \in N). \tag{3.2}$$

Proof. (LF1) is equivalent to the local submodularity:

$$g(p + \lambda\chi_i) + g(p + \mu\chi_j) \geq g(p) + g(p + \lambda\chi_i + \mu\chi_j)$$
$$(p \in \mathbf{R}^n,\ i, j \in N,\ i \neq j,\ \lambda, \mu \geq 0),$$

which can be rewritten as the former property in (3.2); (LF2) is equivalent to the latter in (3.2). □

As is well known, any matrix satisfying (3.2) is positive semidefinite [2,28]. Therefore, quadratic L-convex functions of the form (3.1) are convex functions in the ordinary sense. Noting that the matrix B satisfying (3.2) is singular, i.e., $\mathrm{rank}\, B \leq n - 1$, we denote

$$\mathcal{L} = \bigcup_{n=1}^{\infty} \{B \in \mathcal{S}_n \mid \text{function (3.1) over } \mathbf{R}^n \text{ is L-convex},\ \mathrm{rank}\, B = n - 1\},$$
$$\overline{\mathcal{L}} = \bigcup_{n=1}^{\infty} \{B \in \mathcal{S}_n \mid \text{function (3.1) over } \mathbf{R}^n \text{ is L-convex}\}.$$

We next reveal the block-diagonal structure of matrices in $\overline{\mathcal{L}}$.

Theorem 3.2 *A matrix $B \in \mathcal{S}_n$ belongs to $\overline{\mathcal{L}}$ if and only if B is a block-diagonal matrix w.r.t. some partition of $\{N_k \mid k = 1, 2, \ldots, m\}$ of N such that $B[N_k] \in \mathcal{L}$ $(k = 1, 2, \ldots, m)$, where $m = n - \mathrm{rank}\, B$.*

Here, a block-diagonal matrix B w.r.t. a (sub)partition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N means that all entries in B are equal to zero except for those in principal submatrices $B[N_k]$ $(k = 1, 2, \ldots, m)$.

We then consider an L-convex function of the general form (2.1) involving a subspace K.

Theorem 3.3 *Let $B \in \mathcal{S}_n$, and $K \subseteq \mathbf{R}^n$ be a linear subspace. Then, a quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over K, given by* (2.1)*, is L-convex if and only if $K = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \ldots, d\}$ for some partition $\{N_k \mid k = 1, 2, \ldots, d\}$ of N and the "aggregated" matrix $\widetilde{B} \in \mathcal{S}_d$ defined by $\tilde{b}_{kl} = \sum_{i \in N_k} \sum_{j \in N_l} b_{ij}$ belongs to $\overline{\mathcal{L}}$, where $d = \dim K$.*

3.1.2 L$^\natural$-convex Functions

Based on the definition of L$^\natural$-convexity, Theorems 3.1, 3.2, and 3.3 above for L-convex functions can easily be rephrased for L$^\natural$-convex functions as follows. We denote

$$\mathcal{L}^\natural = \bigcup_{n=1}^{\infty} \{B \in \mathcal{S}_n \mid \text{function (3.1) over } \mathbf{R}^n \text{ is L}^\natural\text{-convex}, \ \mathrm{rank}\, B = n\},$$
$$\overline{\mathcal{L}}^\natural = \bigcup_{n=1}^{\infty} \{B \in \mathcal{S}_n \mid \text{function (3.1) over } \mathbf{R}^n \text{ is L}^\natural\text{-convex}\}.$$

Theorem 3.4 *A quadratic function $g : \mathbf{R}^n \to \mathbf{R}$ given by* (3.1) *with $B \in \mathcal{S}_n$ is L$^\natural$-convex if and only if*

$$b_{ij} \leq 0 \quad (\forall i, j \in N, \ i \neq j), \qquad \sum_{j=1}^{n} b_{ij} \geq 0 \quad (\forall i \in N). \tag{3.3}$$

This theorem shows that the coefficient matrix of a quadratic L$^\natural$-convex function is nothing but a diagonally-dominant symmetric M-matrix. This means, in particular, that a quadratic L$^\natural$-convex function coincides with a Dirichlet form of finite dimension. Note that any symmetric matrix with the property (3.3) is positive semidefinite (cf. [2,28]).

In parallel with Theorems 3.2 and 3.3 we have the following.

Theorem 3.5 *A matrix $B \in \mathcal{S}_n$ belongs to $\overline{\mathcal{L}}^\natural$ if and only if B is a block-diagonal matrix w.r.t. some subpartition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N such that $B[N_k] \in \mathcal{L}$ $(k \in \{1, 2, \ldots, m\})$ and $B[N \setminus \bigcup_{k=1}^m N_k] \in \mathcal{L}^\natural$, where $m = n - \mathrm{rank}\, B$.*

Theorem 3.6 *Let $B \in \mathcal{S}_n$, and $K \subseteq \mathbf{R}^n$ be a linear subspace. Then, a quadratic function g over K, given by* (2.1)*, is L$^\natural$-convex if and only if $K = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \ldots, d\}$ for some subpartition $\{N_k \mid k = 1, 2, \ldots, d\}$ of N and the "aggregated" matrix $\widetilde{B} \in \mathcal{S}_d$ defined by $\widetilde{b}_{kl} = \sum_{i \in N_k} \sum_{j \in N_l} b_{ij}$ belongs to $\overline{\mathcal{L}}^\natural$, where $d = \dim K$.*

3.2.1 M$^\natural$-convex Functions

We first consider a special case where the function f is defined over the entire space $\mathbf{R}^n$:

$$f(x) = \frac{1}{2} x^{\mathrm{T}} A x \qquad (x \in \mathbf{R}^n). \tag{3.4}$$

We denote

$$\mathcal{M}^\natural = \bigcup_{n=1}^{\infty} \{A \in \mathcal{S}_n \mid \text{function (3.4) over } \mathbf{R}^n \text{ is M}^\natural\text{-convex},\ \operatorname{rank} A = n\},$$
$$\overline{\mathcal{M}}^\natural = \bigcup_{n=1}^{\infty} \{A \in \mathcal{S}_n \mid \text{function (3.4) over } \mathbf{R}^n \text{ is M}^\natural\text{-convex}\}.$$

Proposition 3.7
(i) *Any matrix $A \in \mathcal{M}^\natural$ is positive definite.*
(ii) *Any matrix $A \in \overline{\mathcal{M}}^\natural$ is positive semidefinite.*

Let $f : \mathbf{R}^n \to \mathbf{R}$ be of the form (3.4) with a nonsingular $A \in \mathcal{S}_n$. We will show in Theorem 3.8 that M$^\natural$-convexity of such f is characterized by the following properties.

(M$^\natural$-EXC$^+$) $\forall x, y \in \mathbf{R}^n$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) > f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \ (\forall \alpha \in (0, \alpha_0]), \tag{3.5}$$

(M$^\natural$-EXC$^+_{\mathbf{d}}$) $x^{\mathrm{T}} a_i > \min\{0, \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j\} \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x)),$

(M$^\natural$-EXC$^+_{\mathbf{d}}$-R) $x^{\mathrm{T}} a_i > \min\{0, \min_{j \in N \setminus \{i\}} x^{\mathrm{T}} a_j\} \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x)),$

(L$^\natural$-INV) A is nonsingular and A^{-1} satisfies (3.3),

where a_i $(i \in N)$ denotes the i-th column vector of A.

Theorem 3.8 *Let f be a quadratic function over $\mathbf{R}^n$ given by (3.4) with $A \in \mathcal{S}_n$. Then,*

$$A \in \mathcal{M}^\natural \iff (\text{M}^\natural\text{-EXC}^+) \iff (\text{M}^\natural\text{-EXC}^+_{\mathrm{d}}) \iff (\text{M}^\natural\text{-EXC}^+_{\mathrm{d}}\text{-R}) \iff (\text{L}^\natural\text{-INV}).$$

Combining Theorem 3.8 with Theorem 3.4 yields the following relationship between $\mathcal{L}^\natural$ and $\mathcal{M}^\natural$.

Theorem 3.9 *For $B \in \mathcal{L}^\natural$ and $A \in \mathcal{M}^\natural$, we have $B^{-1} \in \mathcal{M}^\natural$ and $A^{-1} \in \mathcal{L}^\natural$. Hence, the inverse-matrix relationship provides a one-to-one correspondence between $\mathcal{L}^\natural$ and $\mathcal{M}^\natural$.*

We then consider the case where the coefficient matrix A is possibly singular. We denote by (M$^\natural$-EXC) the property (M$^\natural$-EXC$^+$) with the strict inequality "$>$" in (3.5) replaced by an inequality "$\geq$". We define (M$^\natural$-EXC$_\mathrm{d}$) from (M$^\natural$-EXC$_\mathrm{d}^+$) in a similar way.

Theorem 3.10 *Let f be a quadratic function over $\mathbf{R}^n$ given by (3.4) with $A \in \mathcal{S}_n$. Then,*

$$A \in \overline{\mathcal{M}}^\natural \iff (\mathrm{M}^\natural\text{-EXC}) \iff (\mathrm{M}^\natural\text{-EXC}_\mathrm{d}).$$

We next reveal the structure of the (possibly singular) coefficient matrix $A \in \mathcal{S}_n$ that defines a quadratic M$^\natural$-convex function over $\mathbf{R}^n$.

Theorem 3.11 *A matrix $A \in \mathcal{S}_n$ belongs to $\overline{\mathcal{M}}^\natural$ if and only if there exist a subpartition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N and an $m \times m$ matrix $\widetilde{A} \in \mathcal{M}^\natural$ such that*

$$a_{ij} = \begin{cases} \tilde{a}_{kl} \ (i \in N_k,\ j \in N_l,\ k, l \in \{1, 2, \ldots, m\}), \\ 0 \quad (\textit{otherwise}), \end{cases}$$

where $m = \operatorname{rank} A$.

We finally consider the general form (2.2) involving a subspace H. We denote by $\overline{\mathcal{M}}$ the set of symmetric matrices A such that the quadratic function of the following form is M-convex:

$$f(x) = \begin{cases} \dfrac{1}{2} x^\mathrm{T} A x \ (x \in \mathbf{R}^n,\ x(N) = 0), \\ +\infty \qquad (\text{otherwise}). \end{cases} \tag{3.6}$$

Theorem 3.12 *Let $A \in \mathcal{S}_n$, and $H \subseteq \mathbf{R}^n$ be a linear subspace. Then, a quadratic function f over H, given by (2.2), is M$^\natural$-convex if and only if there exists a subpartition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N such that*

$$H = \{x \in \mathbf{R}^n \mid x(N_k) = 0 \ (k = 1, 2, \ldots, m)\}, \tag{3.7}$$
$$A[N_k] \in \overline{\mathcal{M}} \quad (k = 1, 2, \ldots, m), \qquad A[N_{m+1}] \in \overline{\mathcal{M}}^\natural,$$

$$a_{ij} - a_{i'j} = a_{ij'} - a_{i'j'} \qquad (i, i' \in N_k,\ j, j' \in N_l,\ k, l \in \{1, 2, \ldots, m\},\ k \neq l), \tag{3.8}$$

$$a_{ij} = a_{ij'} \qquad (i \in N_{m+1},\ j, j' \in N_k,\ k \in \{1, 2, \ldots, m\}), \tag{3.9}$$

where $N_{m+1} = N \setminus \bigcup_{k=1}^{m} N_k$ *and* $m = n - \dim H$.

Remark 3.13 If the subspace H is given by (3.7) and the matrix $A \in \mathcal{S}_n$ satisfies (3.8) and (3.9), then the quadratic function f given by (2.2) satisfies

$$f(x) = \sum_{k=1}^{m+1} \frac{1}{2} x[N_k]^{\mathrm{T}} A[N_k]\, x[N_k] \qquad (x \in H). \tag{3.10}$$

Thus, the coefficient matrix A is effectively block-diagonal. □

3.2.2 M-convex Functions

For any matrix $A \in \mathcal{S}_n$, we define $A^\natural = (a^\natural_{ij}) \in \mathcal{S}_{n-1}$ by

$$a^\natural_{ij} = a_{ij} - a_{in} - a_{nj} + a_{nn} \qquad (i, j \in N \setminus \{n\}).$$

We denote

$$\mathcal{M} = \bigcup_{n=1}^{\infty} \{A \in \mathcal{S}_n \mid \text{function (3.6) is M-convex},\ \mathrm{rank}\, A^\natural = n - 1\},$$

$$\overline{\mathcal{M}} = \bigcup_{n=1}^{\infty} \{A \in \mathcal{S}_n \mid \text{function (3.6) is M-convex}\}.$$

It is easy to see from the definition of M$^\natural$-convexity that

$$A \in \mathcal{M} \iff A^\natural \in \mathcal{M}^\natural, \qquad A \in \overline{\mathcal{M}} \iff A^\natural \in \overline{\mathcal{M}}^\natural. \tag{3.11}$$

Note also that for a matrix $\widehat{A} = (\widehat{a}_{ij}) \in \mathcal{S}_{n+1}$ given by

$$\widehat{a}_{ij} = a_{ij} \quad \text{if } i, j \in N, \qquad \widehat{a}_{ij} = 0 \quad \text{otherwise}, \tag{3.12}$$

we have

$$A \in \mathcal{M}^\natural \iff \widehat{A} \in \mathcal{M}, \qquad A \in \overline{\mathcal{M}}^\natural \iff \widehat{A} \in \overline{\mathcal{M}}. \tag{3.13}$$

For a quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ of the form (3.6), consider the following properties:

(M-EXC$^+$) $\forall x, y \in \mathbf{R}^n$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) > f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \quad (\forall \alpha \in (0, \alpha_0]),$$

$$\textbf{(M-EXC}^{+}_{\mathbf{d}}\textbf{)} \quad x^{\mathrm{T}} a_i > \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x)).$$

We denote by (M-EXC$_{\mathrm{d}}$) the property (M-EXC$^+_{\mathrm{d}}$) with the strict inequality "$>$" replaced by an inequality "$\geq$".

Theorem 3.14 *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a quadratic function given by* (3.6) *with $A \in \mathcal{S}_n$. Then,*

$$\text{(i) } A \in \mathcal{M} \iff \text{(M-EXC}^+\text{)} \iff \text{(M-EXC}^+_{\mathrm{d}}\text{)},$$
$$\text{(ii) } A \in \overline{\mathcal{M}} \iff \text{(M-EXC}_{\mathrm{d}}\text{)}.$$

We next reveal the structure of the (possibly singular) coefficient matrix $A \in \mathcal{S}_n$ that defines a quadratic M-convex function over $\mathbf{R}^n$.

Theorem 3.15 *A matrix $A \in \mathcal{S}_n$ belongs to $\overline{\mathcal{M}}$ if and only if there exist a partition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N and an $m \times m$ matrix $\widetilde{A} \in \mathcal{M}$ such that $a_{ij} = \widetilde{a}_{kl}$ $(i \in N_k,\ j \in N_l,\ k, l \in \{1, 2, \ldots, m\})$, where $m = \mathrm{rank}\, A$.*

We finally consider the structure of a quadratic M-convex function of the general form (2.2).

Theorem 3.16 *Let $A \in \mathcal{S}_n$, and $H \subseteq \mathbf{R}^n$ be a linear subspace. Then, a quadratic function f over H, given by* (2.2)*, is M-convex if and only if there exists a partition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N such that* (3.7)*, $A[N_k] \in \overline{\mathcal{M}}$ $(k = 1, 2, \ldots, m)$, and* (3.8)*, where $m = n - \dim H$.*

4 Conjugacy Between Quadratic L-convex and M-convex Functions

The conjugacy relationship between quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions is investigated in this section. For a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, we define $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f^\bullet(p) = \sup\{p^{\mathrm{T}} x - f(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n).$$

The transformation from f to $f^\bullet$ is called the *Legendre–Fenchel transformation*, and the function $f^\bullet$ is the *conjugate* of f. We have $(f^\bullet)^\bullet = f$ for a closed convex function. See [24,27] for details on the conjugacy of convex functions.

Suppose that f is a quadratic convex function over H given by (2.2) with a positive semidefinite matrix $A \in \mathcal{S}_n$ and a linear subspace $H \subseteq \mathbf{R}^n$. Then, the conjugate $g = f^\bullet$ is a quadratic convex function given by (2.1) with a positive semidefinite matrix $B \in \mathcal{S}_n$ and a linear subspace $K \subseteq \mathbf{R}^n$ such that $K = (H \cap \ker A)^\perp$ and $H = (K \cap \ker B)^\perp$, where

$$\begin{aligned} X^\perp &= \{p \in \mathbf{R}^n \mid p^{\mathrm{T}} x = 0 \ (\forall x \in X)\} \qquad (X \subseteq \mathbf{R}^n), \\ \ker A &= \{x \in \mathbf{R}^n \mid Ax = 0\} \qquad (A \in \mathcal{S}_n). \end{aligned}$$

We may write this also as

$$\mathrm{dom}\, f^\bullet = (\arg\min f)^\perp, \qquad \mathrm{dom}\, f = (\arg\min f^\bullet)^\perp, \tag{4.1}$$

where $\arg\min f$ denotes the set of minimizers of f. In particular, if $H = \mathbf{R}^n$ and A is positive definite, then the conjugate $f^\bullet$ is explicitly written as

$$f^\bullet(p) = \frac{1}{2} p^{\mathrm{T}} A^{-1} p \qquad (p \in \mathbf{R}^n). \tag{4.2}$$

4.1 Conjugacy Between L-convexity and M-convexity

We show that quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions are conjugate to each other under the Legendre–Fenchel transformation.

Theorem 4.1
(i) *Let $B \in \mathcal{S}_n$, and $K \subseteq \mathbf{R}^n$ be a linear subspace. If a quadratic function g over K, given by* (2.1)*, is L-convex (resp. L$^\natural$-convex), then the conjugate $g^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is a quadratic M-convex (resp. M$^\natural$-convex) function.*
(ii) *Let $A \in \mathcal{S}_n$, and $H \subseteq \mathbf{R}^n$ be a linear subspace. If a quadratic function f over H, given by* (2.2)*, is M-convex (resp. M$^\natural$-convex), then the conjugate $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is a quadratic L-convex (resp. L$^\natural$-convex) function.*

Note that we have already seen the essence of the above conjugacy in Theorem 3.9 and (4.2). The former reveals the inverse-matrix relationship between the coefficient matrices of quadratic L$^\natural$-convex and M$^\natural$-convex functions, and the latter shows a fundamental fact that the coefficient matrices of a conjugate pair of quadratic convex functions are the inverse of each other. The proof of Theorem 4.1, to be given below, is a rather straightforward argument to build up the conjugacy in this basic case by means of the structure theorems such as

Theorems 3.6 and 3.12 for the general case. We also use the following (easy) fact, indicating the conjugacy between L-/L$^\natural$-convex and M-/M$^\natural$-convex linear subspaces.

Proposition 4.2 *For a pair of linear subspaces* $K = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \ldots, m\}$ *and* $H = \{x \in \mathbf{R}^n \mid x(N_k) = 0 \ (k = 1, 2, \ldots, m)\}$ *defined by a subpartition* $\{N_k \mid k = 1, 2, \ldots, m\}$ *of* N*, we have* $K = H^\perp$*.*

We now prove Theorem 4.1 in the case of L-convex g and M-convex f.

Proof of (i). [Case 1: $K = \mathbf{R}^n$, $B \in \mathcal{L}$] Since $\arg\min g = \{\lambda \mathbf{1} \mid \lambda \in \mathbf{R}\}$, we have $\mathrm{dom}\, g^\bullet = \{x \in \mathbf{R}^n \mid x(N) = 0\}$ by (4.1) and Proposition 4.2. The property (LF2) also implies $g^\bullet(x) = (1/2)x^{\mathrm{T}}Ax$ ($x \in \mathrm{dom}\, g^\bullet$), where $A \in \mathcal{S}_n$ is given by $A[N \setminus \{n\}] = (B[N \setminus \{n\}])^{-1}$ and $a_{in} = a_{ni} = 0$ ($i \in N$). Since $B[N \setminus \{n\}] \in \mathcal{L}^\natural$, we have $A \in \overline{\mathcal{M}}$ by Theorem 3.9 and (3.13), which yields the M-convexity of $g^\bullet$.

[Case 2: $K = \mathbf{R}^n$, $B \in \overline{\mathcal{L}}$] By Theorem 3.2, there exists a partition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N such that $\arg\min g = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \ldots, m\}$ and $g(p) = \sum_{k=1}^m g_k(p[N_k])$ ($p \in \mathbf{R}^n$), where $g_k(p[N_k]) = (1/2)p[N_k]^{\mathrm{T}}\, B[N_k]\, p[N_k]$ and $B[N_k] \in \mathcal{L}$. This implies that each $g_k^\bullet$ is M-convex by Case 1 and $g^\bullet(x) = \sum_{k=1}^m g_k^\bullet(x[N_k])$. Hence, $g^\bullet$ is also an M-convex function.

[Case 3: general K] By Theorem 3.3, there exists a partition $\{N_k \mid k = 1, 2, \ldots, d\}$ of N such that $K = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \ldots, d\}$ and the matrix $\widetilde{B} \in \mathcal{S}_d$ defined by $\widetilde{b}_{kl} = \sum_{i \in N_k} \sum_{j \in N_l} b_{ij}$ belongs to $\overline{\mathcal{L}}$. Define $\widetilde{g} : \mathbf{R}^d \to \mathbf{R}$ by $\widetilde{g}(q) = (1/2)q^{\mathrm{T}}\widetilde{B}q$ ($q \in \mathbf{R}^d$). Then, we have (M-EXC) for $(\widetilde{g})^\bullet$ by Case 2, and $g^\bullet(x) = (\widetilde{g})^\bullet(\widetilde{x})$ for any $x \in \mathbf{R}^n$ and $\widetilde{x} \in \mathbf{R}^d$ with $\widetilde{x}_k = x(N_k)$ ($k = 1, \ldots, d$). Hence, $g^\bullet$ also satisfies (M-EXC), i.e., $g^\bullet$ is M-convex. □

Proof of (ii). [Case 1: $H = \{x \in \mathbf{R}^n \mid x(N) = 0\}$] We may assume that $A \in \overline{\mathcal{M}}$ is given as $A[N \setminus \{n\}] = A' \in \overline{\mathcal{M}}^\natural$ and $a_{in} = a_{ni} = 0$ ($i \in N$) (cf. (3.11), (3.13)). In the following, we first consider the case where A' is nonsingular, and then the general case.

[Case 1-1: $\mathrm{rank}\, A' = n - 1$] Since $\arg\min f = \{\mathbf{0}\}$ and $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid x(N) = 0\}$, we have $\mathrm{dom}\, f^\bullet = \mathbf{R}^n$ by (4.1) and $f^\bullet(p) = (1/2)p^{\mathrm{T}}Bp$ for $p \in \mathrm{dom}\, f^\bullet$, where B is given by

$$B = \left[\begin{array}{c|c} (A')^{-1} & -(A')^{-1}\mathbf{1}' \\ \hline -(\mathbf{1}')^{\mathrm{T}}(A')^{-1} & (\mathbf{1}')^{\mathrm{T}}(A')^{-1}\mathbf{1}' \end{array}\right]$$

and $\mathbf{1}' \in \mathbf{R}^{n-1}$ is the vector with all components equal to one. We have $B \in \mathcal{L}$ since $(A')^{-1} \in \mathcal{L}^\natural$ by Theorem 3.9. Hence, $f^\bullet$ is L-convex.

[Case 1-2: general A'] By Theorem 3.15, there exists a partition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N such that $\arg\min f = \{x \in \mathbf{R}^n \mid x(N_k) = 0 \ (k = 1, 2, \ldots, m)\}$ and $a_{ij} = \widetilde{a}_{kl}$ $(i \in N_k,\ j \in N_l,\ k, l \in \{1, 2, \ldots, m\})$ for some $m \times m$ matrix $\widetilde{A} \in \mathcal{M}$. By (4.1) and Proposition 4.2, we have $\mathrm{dom}\, f^\bullet = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \ldots, m\}$. Define $\widetilde{f} : \mathbf{R}^m \to \mathbf{R} \cup \{+\infty\}$ by

$$\widetilde{f}(y) = \begin{cases} \dfrac{1}{2} y^{\mathrm{T}} \widetilde{A} y \ \ (y \in \mathbf{R}^m,\ \displaystyle\sum_{k=1}^{m} y_k = 0), \\ +\infty \qquad (\text{otherwise}). \end{cases}$$

Then, $(\widetilde{f})^\bullet$ is L-convex by Case 1-1. Since $f^\bullet(\sum_{k=1}^{m} q_k \chi_{N_k}) = (\widetilde{f})^\bullet(q)$ for $q \in \mathbf{R}^d$, $f^\bullet$ is also L-convex.

[Case 2: general H] By Theorem 3.16, there exists a partition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N such that $f(x) = \sum_{k=1}^{m} f_k(x[N_k])$ $(x \in \mathbf{R}^n)$ with a family of quadratic M-convex functions $f_k : \mathbf{R}^{N_k} \to \mathbf{R} \cup \{+\infty\}$ given by

$$f_k(x[N_k]) = \begin{cases} \dfrac{1}{2} x[N_k]^{\mathrm{T}} A[N_k] x[N_k] \ \ (x[N_k] \in \mathbf{R}^{N_k},\ \displaystyle\sum_{i \in N_k} x_i = 0), \\ +\infty \qquad\qquad\qquad (\text{otherwise}). \end{cases}$$

Since each $(f_k)^\bullet$ is L-convex by Case 1-2, $f^\bullet(p) = \sum_{k=1}^{m} (f_k)^\bullet(p[N_k])$ $(p \in \mathbf{R}^n)$ is also L-convex. □

5 Quadratic L-convex/M-convex Functions over the Integer Lattice

In this section, we consider discrete functions defined over the integer lattice $\mathbf{Z}^n$ to indicate a subtle difference between the discrete case and the continuous case. Transition to the discrete case is quite smooth for L-convexity (Theorem 5.1) but that for M-convexity is something unexpected (see Theorems 5.2, 5.3, and 5.4).

5.1 L-convex/L$^\natural$-convex Functions

For a symmetric matrix $B \in \mathcal{S}_n$, we define a quadratic function $g : \mathbf{Z}^n \to \mathbf{R}$ by $g(p) = (1/2)p^{\mathrm{T}} B p$ $(p \in \mathbf{Z}^n)$. We call such a function *L-convex* if it satisfies the properties (LF1[**Z**]) and (LF2[**Z**]):

(LF1[Z]) $\quad g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}_{\mathbf{Z}}\, g),$
(LF2[Z]) $\quad \exists r \in \mathbf{R}$ such that $g(p+\lambda \mathbf{1}) = g(p)+\lambda r \quad (\forall p \in \mathrm{dom}_{\mathbf{Z}}\, g,\ \lambda \in \mathbf{Z}),$

where $\mathrm{dom}_{\mathbf{Z}}\, g = \{p \in \mathbf{Z}^n \mid g(p) < +\infty\}$; g is called *L♮-convex* if the function $\widehat{g} : \mathbf{Z}^{\widehat{N}} \to \mathbf{R}$, where $\widehat{N} = \{0\} \cup N$, defined by $\widehat{g}(p_0, p) = g(p - p_0\mathbf{1})$ ($(p_0, p) \in \mathbf{Z}^{\widehat{N}}$) is L-convex. We denote by $\overline{\mathcal{L}}[\mathbf{Z}]$ (resp., $\overline{\mathcal{L}}^{\natural}[\mathbf{Z}]$) the set of matrices $B \in \mathcal{S}_n$ such that the function g over $\mathbf{Z}^n$ is L-convex (resp., L♮-convex). L-/L♮-convexity of quadratic functions over $\mathbf{Z}^n$ can be characterized by the same properties as those over $\mathbf{R}^n$, i.e., $\overline{\mathcal{L}}[\mathbf{Z}] = \overline{\mathcal{L}}$ and $\overline{\mathcal{L}}^{\natural}[\mathbf{Z}] = \overline{\mathcal{L}}^{\natural}$. Proofs are quite similar and therefore omitted.

Theorem 5.1 *Let $B \in \mathcal{S}_n$. Then,*

$$\text{(i) } B \in \overline{\mathcal{L}}[\mathbf{Z}] \iff (3.2), \qquad \text{(ii) } B \in \overline{\mathcal{L}}^{\natural}[\mathbf{Z}] \iff (3.3).$$

5.2 M-convex/M♮-convex Functions

For a symmetric matrix $A \in \mathcal{S}_n$, we define a quadratic function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ by

$$f(x) = \begin{cases} \dfrac{1}{2} x^{\mathrm{T}} A x & (x \in \mathbf{Z}^n,\ x(N) = 0), \\ +\infty & \text{(otherwise)}. \end{cases} \tag{5.1}$$

We call such a function *M-convex* if it satisfies the property (M-EXC[**Z**]):

(M-EXC[Z]) $\forall x, y \in \mathrm{dom}_{\mathbf{Z}}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$:

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

We denote by $\overline{\mathcal{M}}[\mathbf{Z}]$ the set of matrices $A \in \mathcal{S}_n$ such that the function (5.1) over $\{x \in \mathbf{Z}^n \mid x(N) = 0\}$ is M-convex. Matrices in $\overline{\mathcal{M}}[\mathbf{Z}]$ can be characterized as follows:

Theorem 5.2 *A matrix $A \in \mathcal{S}_n$ belongs to $\overline{\mathcal{M}}[\mathbf{Z}]$ if and only if*

$$a_{ij} + a_{kl} \geq \min\{a_{ik} + a_{jl}, a_{il} + a_{jk}\} \qquad (\forall i, j, k, l \in N \text{ with } \{i, j\} \cap \{k, l\} = \emptyset). \tag{5.2}$$

Proof. A function f given by (5.1) is M-convex if (M-EXC[**Z**]) holds for any $x, y \in \mathrm{dom}_{\mathbf{Z}}\, f$ with $\sum_{k=1}^{n} |x_k - y_k| = 4$ (cf. [17, Th. 3.1]), which can be simply rewritten as (5.2). □

A quadratic function $f : \mathbf{Z}^n \to \mathbf{R}$ given by

$$f(x) = \frac{1}{2} x^{\mathrm{T}} A x \qquad (x \in \mathbf{Z}^n) \tag{5.3}$$

is called M$^\natural$-*convex* if the function $\widehat{f} : \mathbf{Z}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$, where $\widehat{N} = \{0\} \cup N$, defined by

$$\widehat{f}(x_0, x) = \begin{cases} f(x) \ (x_0 = -x(N)), \\ +\infty \ \ (\text{otherwise}), \end{cases} \qquad ((x_0, x) \in \mathbf{Z}^{\widehat{N}})$$

is M-convex. We denote by $\overline{\mathcal{M}}^\natural[\mathbf{Z}]$ the set of matrices $A \in \mathcal{S}_n$ such that the function (5.3) over $\mathbf{Z}^n$ is M$^\natural$-convex. Theorem 5.2 gives the following characterization of quadratic M$^\natural$-convex functions.

Theorem 5.3 *A matrix $A \in \mathcal{S}_n$ belongs to $\overline{\mathcal{M}}^\natural[\mathbf{Z}]$ if and only if* (5.2), $a_{ij} \geq 0$ $(\forall i, j \in N)$, *and* $a_{ij} \geq \min\{a_{ik}, a_{jk}\}$ $(\forall i, j, k \in N$ *with* $k \notin \{i, j\})$ *hold.*

Theorem 5.4 $\overline{\mathcal{M}}[\mathbf{Z}] \subseteq \overline{\mathcal{M}}$ *and* $\overline{\mathcal{M}}^\natural[\mathbf{Z}] \subseteq \overline{\mathcal{M}}^\natural$.

Proof. We show the former only. (M-EXC[$\mathbf{Z}$]) for f of (5.1) can be rewritten as

$$\min_{j \in \mathrm{supp}^-(x)} \{x^{\mathrm{T}} a_j - x^{\mathrm{T}} a_i + a_{ii} + a_{jj} - 2a_{ij}\} \leq 0$$
$$(\forall x \in \mathbf{Z}^n \text{ with } x(N) = 0, \ \forall i \in \mathrm{supp}^+(x)),$$

which, together with Theorem 5.2, implies $\min_{j \in \mathrm{supp}^-(x)} \{x^{\mathrm{T}} a_j - x^{\mathrm{T}} a_i\} \leq 0$ for any $x \in \mathbf{Z}^n$ with $x(N) = 0$ and any $i \in \mathrm{supp}^+(x)$. Hence, (M-EXC$_\mathrm{d}$) holds for A, i.e., $A \in \overline{\mathcal{M}}$ by Theorem 3.14 (ii). □

The inclusions in Theorem 5.4 are proper; for example, the symmetric matrix $A \in \mathcal{S}_3$ given by

$$A = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 5 & 3 \\ 4 & 3 & 6 \end{bmatrix}, \qquad \text{where} \qquad A^{-1} = \frac{1}{7} \begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & -1 \\ -2 & -1 & 3 \end{bmatrix},$$

belongs to $\overline{\mathcal{M}}^\natural$ and not to $\overline{\mathcal{M}}^\natural[\mathbf{Z}]$.

6 Proofs

6.1 Proofs for Quadratic L-convex/L$^\natural$-convex Functions

Proof of Theorem 3.2. The "if" part is immediate from Theorem 3.1. To prove the "only if" part, assume $B \in \overline{\mathcal{L}}$. Then, B satisfies the condition (3.2) by Theorem 3.1. Since $\arg\min g$ is an L-convex linear subspace (cf. [21, Th. 4.34]), there exists a partition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N such that $\arg\min g = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \ldots, m\}$ by Proposition 2.1. Since $\ker B = \arg\min g$, we have $b_i^{\mathrm{T}} \chi_{N_k} = \sum_{j \in N_k} b_{ij} = 0$ for all $i \in N$ and $k = 1, 2, \ldots, m$, where b_i is the i-th column of B. This property, together with (3.2), implies that B is a block-diagonal matrix such that each block $B[N_k]$ belongs to $\mathcal{L}$. It should be noted that $\mathrm{rank}\, B[N_k] = |N_k| - 1$ follows from the equation $\ker B[N_k] = \chi_{N_k}$. □

Proof of Theorem 3.3. We first prove the "only if" part. Suppose that g is L-convex. Then, $\mathrm{dom}\, g = K$ is an L-convex linear subspace and can be represented as $K = \mathrm{span}\{\chi_{N_k} \mid k = 1, \ldots, d\}$ by some partition $\{N_k \mid k = 1, \ldots, d\}$ of N. Define $\widetilde{g} : \mathbf{R}^d \to \mathbf{R}$ by $\widetilde{g}(q) = (1/2) q^{\mathrm{T}} \widetilde{B} q$ $(q \in \mathbf{R}^d)$, where $\widetilde{b}_{kl} = \sum_{i \in N_k} \sum_{j \in N_l} b_{ij}$. Then, we have $g(\sum_{k=1}^{d} q_k \chi_{N_k}) = \widetilde{g}(q)$ $(\forall q \in \mathbf{R}^d)$, implying L-convexity of $\widetilde{g}$, i.e., $\widetilde{B} \in \overline{\mathcal{L}}$. The "if" part is obvious from the discussion above and Proposition 2.1. □

6.2 Proofs for Quadratic M-convex/M$^\natural$-convex Functions

6.2.1 Proof of Proposition 3.7

It can be easily shown that $A \in \mathcal{M}^\natural$ (resp., $A \in \overline{\mathcal{M}^\natural}$) implies (M$^\natural$-EXC$_{\mathrm{d}}^{+}$-R) (resp., (M$^\natural$-EXC$_{\mathrm{d}}$)) (see the proofs of Theorems 3.8 and 3.10 below). Hence, it suffices to prove the following.

Proposition 6.1
(i) *Any matrix* $A \in \mathcal{S}_n$ *satisfying* (M$^\natural$-EXC$_{\mathrm{d}}^{+}$-R) *is positive definite.*
(ii) *Any matrix* $A \in \mathcal{S}_n$ *satisfying* (M$^\natural$-EXC$_{\mathrm{d}}$) *is positive semidefinite.*

Proof. We prove (i) only since (ii) can be shown similarly. Let λ be an eigenvalue of A, and x be the corresponding eigenvector with $\mathrm{supp}^+(x) \neq \emptyset$. Let $i = i_*$ minimize the value λx_i among $i \in \mathrm{supp}^+(x)$. (M$^\natural$-EXC$_{\mathrm{d}}^{+}$-R) implies

$$\lambda x_{i_*} = x^{\mathrm{T}} a_{i_*} > \min\{0, \min_{j \in N \setminus \{i_*\}} x^{\mathrm{T}} a_j\} = \min\{0, \min_{j \in N \setminus \{i_*\}} \lambda x_j\}. \tag{6.1}$$

By the choice of i_*, we have $\lambda x_{i_*} \le \lambda x_j$ for any $j \in \mathrm{supp}^+(x)$. Hence, (6.1) implies $\lambda x_{i_*} > 0$ or $\lambda x_{i_*} > \lambda x_j$ for some $j \in \mathrm{supp}^-(x)$, each of which yields $\lambda > 0$. Hence, A is positive definite. □

6.2.2 Proofs of Theorems 3.8 and 3.14 (i)

We first prove the equivalence "(M$^\natural$-EXC) and $\mathrm{rank}\, A = n \iff$ (M$^\natural$-EXC$^+$) $\iff$ (M$^\natural$-EXC$^+_\mathrm{d}$) $\iff$ (M$^\natural$-EXC$^+_\mathrm{d}$-R) $\iff$ (L$^\natural$-INV)."

Lemma 6.2 (Farkas' lemma) *Let A be an $m \times n$ real matrix and $b \in \mathbf{R}^m$. Then, $Ax = b$ for some nonnegative $x \in \mathbf{R}^n$ if and only if $y^\mathrm{T} b \ge 0$ for every $y \in \mathbf{R}^m$ with $y^\mathrm{T} A \ge \mathbf{0}^\mathrm{T}$.*

Lemma 6.3 (M$^\natural$-EXC$^+_\mathrm{d}$-R) $\iff$ (L$^\natural$-INV).

Proof. We first note that (M$^\natural$-EXC$^+_\mathrm{d}$-R) implies the nonsingularity of A by Proposition 6.1. Denoting A^{-1} by $B = (b_{ij})$, we have $AB = I$, i.e., $\sum_{j=1}^n b_{ji} a_j = \chi_i$ $(i \in N)$, which can be rewritten as

$$\Big(\sum_{j=1}^n b_{ji}\Big) a_i + \sum_{j \ne i} (-b_{ji})(a_i - a_j) = \chi_i \qquad (i \in N).$$

The condition (3.3) for B is equivalent to all the coefficients in this linear combination being nonnegative, whereas the latter condition is equivalent, by Lemma 6.2, to $y^\mathrm{T} a_i > \min\{0, \min_{j \ne i} y^\mathrm{T} a_j\}$ for any $y \in \mathbf{R}^n$ with $y^\mathrm{T}\chi_i = y_i > 0$, which is nothing but (M$^\natural$-EXC$^+_\mathrm{d}$-R). □

To show the equivalence "(M$^\natural$-EXC$^+_\mathrm{d}$) $\iff$ (M$^\natural$-EXC$^+_\mathrm{d}$-R)," we use the following property:

Lemma 6.4 *If $A \in \mathcal{S}_n$ satisfies* (L$^\natural$-INV)*, then any principal submatrix of A satisfies* (L$^\natural$-INV).

Proof. Put $B = A^{-1}$. Then, B satisfies (3.3). Partition A and B as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

with submatrices $A_{11}, B_{11} \in \mathcal{S}_{n-1}$, $A_{12}^\mathrm{T} = A_{21} = (a_{n1}, a_{n2}, \ldots, a_{n,n-1})$, $B_{12}^\mathrm{T} = B_{21} = (b_{n1}, b_{n2}, \ldots, b_{n,n-1})$, $A_{22} = a_{nn}$, $B_{22} = b_{nn}$. We will show that A_{11} is nonsingular and $(A_{11})^{-1}$ satisfies (3.3).

We have $b_{nn} > 0$ since B is a nonsingular matrix with (3.3). Therefore, the inverse of A_{11} is given by $\widehat{B} = B_{11} - (1/b_{nn})B_{12}B_{21}$, which shows that A_{11} is nonsingular. For any $i, j \in N\setminus\{n\}$ with $i \neq j$, we have $\widehat{b}_{ij} = b_{ij} - b_{in}b_{nj}/b_{nn} \leq 0$ since $b_{ij}, b_{in}, b_{nj} \leq 0$ and $b_{nn} > 0$. For any $i \in N \setminus \{n\}$, we have

$$\sum_{j=1}^{n-1} \widehat{b}_{ij} = \sum_{j=1}^{n-1} b_{ij} - (b_{in}/b_{nn}) \sum_{j=1}^{n-1} b_{nj} \geq -b_{in} - (b_{in}/b_{nn}) \sum_{j=1}^{n-1} b_{nj} \geq 0.$$

Hence, $\widehat{B} = (A_{11})^{-1}$ satisfies (3.3). □

Lemma 6.5 (M$^\natural$-EXC$^+_\mathrm{d}$) $\iff$ (M$^\natural$-EXC$^+_\mathrm{d}$-R).

Proof. It suffices to show the "$\Longleftarrow$" direction. Assume (M$^\natural$-EXC$^+_\mathrm{d}$-R) for A. For $x \in \mathbf{R}^n$ it suffices to consider $i \in \mathrm{supp}^+(x)$ with minimum $x^\mathrm{T}a_i$. Put $S = \{j \in N \mid x_j \neq 0\}$, $\overline{A} = (\overline{a}_j \mid j \in S) = A[S]$, and $\overline{x} = x[S]$. Then, we have $i \in \mathrm{supp}^+(\overline{x}) = \mathrm{supp}^+(x)$, $\mathrm{supp}^-(\overline{x}) = \mathrm{supp}^-(x)$, and $\overline{x}^\mathrm{T}\overline{a}_j = x^\mathrm{T}a_j$ $(\forall j \in S)$. By Lemmas 6.3 and 6.4, we have (M$^\natural$-EXC$^+_\mathrm{d}$-R) for $\overline{A}$, implying

$$\overline{x}^\mathrm{T}\overline{a}_i > \min\{0, \min_{j\in S\setminus\{i_*\}} \overline{x}^\mathrm{T}\overline{a}_j\}. \tag{6.2}$$

By the choice of i, we have $\overline{x}^\mathrm{T}\overline{a}_i \leq \overline{x}^\mathrm{T}\overline{a}_j$ for $j \in \mathrm{supp}^+(\overline{x})$. Hence, from (6.2) follows $\overline{x}^\mathrm{T}\overline{a}_i > \min\{0, \min_{j\in\mathrm{supp}^-(\overline{x})} \overline{x}^\mathrm{T}\overline{a}_j\}$. □

Lemma 6.6 (M$^\natural$-EXC$^+$) $\iff$ (M$^\natural$-EXC$^+_\mathrm{d}$).

Proof. Since

$$\begin{aligned} &f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) - f(x) - f(y) \\ &= \alpha(x-y)^\mathrm{T}(a_j - a_i) + \alpha^2(a_{ii} + a_{jj} - 2a_{ij}), \end{aligned}$$

(M$^\natural$-EXC$^+$) for f can be rewritten as follows:

$$\begin{aligned} &\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x),\ \exists j \in \mathrm{supp}^-(x) \cup \{0\}: \\ &\qquad [x^\mathrm{T}a_j < x^\mathrm{T}a_i] \text{ or } [x^\mathrm{T}a_j = x^\mathrm{T}a_i \text{ and } a_{ii} + a_{jj} - 2a_{ij} < 0], \end{aligned} \tag{6.3}$$

where $a_0 = \mathbf{0}$ and $a_{i0} = a_{0i} = 0$ $(i \in N)$ by convention. In particular, (6.3) for $x = \chi_i - \chi_k$ and $x = \chi_k - \chi_i$ implies $a_{ii} + a_{kk} - 2a_{ik} \geq 0$ for all $i \in N$ and $k \in N \cup \{0\}$. Hence the second case in (6.3) is impossible, and (6.3) is equivalent to (M$^\natural$-EXC$^+_\mathrm{d}$). □

Lemma 6.7 (M$^\natural$-EXC) *and* $\mathrm{rank}\, A = n$ $\iff$ (M$^\natural$-EXC$^+$).

Proof. Assume (M$^\natural$-EXC$^+$) for f. Then, we can show that $\mathrm{rank}\, A = n$ in the same way as Proposition 6.1 (i), which proves the "$\Longleftarrow$" direction. On the other hand, (M$^\natural$-EXC) for f yields the positive semidefiniteness of A, which can be shown similarly to Proposition 6.1 (ii). This, together with rank $A = n$, implies the strict convexity for f. Hence, (M$^\natural$-EXC$^+$) follows. □

The equivalence "$A \in \mathcal{M} \iff$ (M-EXC$^+$) $\iff$ (M-EXC$^+_\mathrm{d}$)" can be shown in the similar way as Lemmas 6.6 and 6.7, implying Theorem 3.14 (i).

Define $\widehat{A} = (\widehat{a}_{ij}) \in \mathcal{S}_{n+1}$ by (3.12). By (3.13) and Theorem 3.14 (i), we have the equivalence "$A \in \mathcal{M}^\natural \iff \widehat{A} \in \mathcal{M} \iff$ (M-EXC$^+_\mathrm{d}$) for $\widehat{A}$", where the last condition can be rewritten in terms of A as

$$
\textbf{(M}^\natural\textbf{-PRJ}^+_\mathbf{d}\textbf{)}
\begin{cases}
\text{(i)} \;\; x^\mathrm{T} a_i > \min\{0, \displaystyle\min_{j \in \mathrm{supp}^-(x)} x^\mathrm{T} a_j\} \\
\qquad\qquad (\forall x \in \mathbf{R}^n \text{ with } x(N) > 0,\ \forall i \in \mathrm{supp}^+(x)), \\
\text{(ii)} \;\; x^\mathrm{T} a_i > \displaystyle\min_{j \in \mathrm{supp}^-(x)} x^\mathrm{T} a_j \\
\qquad\qquad (\forall x \in \mathbf{R}^n \text{ with } x(N) \le 0,\ \forall i \in \mathrm{supp}^+(x)), \\
\text{(iii)} \; 0 > \displaystyle\min_{j \in \mathrm{supp}^-(x)} x^\mathrm{T} a_j \quad (\forall x \in \mathbf{R}^n \text{ with } x(N) < 0).
\end{cases}
$$

To conclude the proof of Theorem 3.8, we show the equivalence "(M$^\natural$-PRJ$^+_\mathrm{d}$) $\iff$ (M$^\natural$-EXC$^+_\mathrm{d}$)."

Lemma 6.8 *If $A \in \mathcal{S}_n$ satisfies* (L$^\natural$-INV) *and if $x(N) < 0$, then $0 > \min_{j \in N} x^\mathrm{T} a_j$.*

Proof. Put $B = A^{-1}$. Then we have

$$
0 > x(N) = x^\mathrm{T}\mathbf{1} = x^\mathrm{T} AB\mathbf{1} = \sum_{j=1}^{n} (x^\mathrm{T} a_j)(\sum_{i=1}^{n} b_{ji}),
$$

whereas $\sum_{i=1}^{n} b_{ji} \ge 0$ for every $j \in N$ by (3.3). Hence $x^\mathrm{T} a_j < 0$ for some $j \in N$. □

Lemma 6.9 *If $A \in \mathcal{S}_n$ satisfies* (M$^\natural$-EXC$_\mathrm{d}$), *then $a_{ii} \ge a_{ij} \ge 0$ ($\forall i, j \in N$).*

Proof. For a sufficiently small $\varepsilon > 0$, (M$^\natural$-EXC$_\mathrm{d}$) with $x = \varepsilon\chi_j + \chi_i$ implies $a_{ij} \ge 0$, whereas (M$^\natural$-EXC$_\mathrm{d}$) with $x = \varepsilon\chi_j - \chi_i$ implies $a_{ii} \ge a_{ij}$. □

Lemma 6.10 (M$^\natural$-PRJ$^+_\mathrm{d}$) $\iff$ (M$^\natural$-EXC$^+_\mathrm{d}$).

Proof. We assume (M$^\natural$-EXC$^+_\mathrm{d}$) for A, and prove (M$^\natural$-PRJ$^+_\mathrm{d}$). The property (i) in (M$^\natural$-PRJ$^+_\mathrm{d}$) is immediate from (M$^\natural$-EXC$^+_\mathrm{d}$). In the same way as Lemma 6.5, we can show by using Lemma 6.8 that

$$
0 > \min\{x^\mathrm{T} a_j \mid j \in N,\ x_j \ne 0\} \qquad (x \in \mathbf{R}^n \text{ with } x(N) < 0),
$$

which, together with ($\mathrm{M}^\natural$-$\mathrm{EXC}_\mathrm{d}^+$), implies (iii).

We now prove (ii). Assume $x(N) \leq 0$ and $i \in \mathrm{supp}^+(x)$.

[Case 1: $x(N) < 0$] If $x^\mathrm{T} a_i \leq 0$, then ($\mathrm{M}^\natural$-$\mathrm{EXC}_\mathrm{d}^+$) immediately yields (ii). Otherwise, we have $x^\mathrm{T} a_i > 0$, and therefore the property (iii) shown above implies $x^\mathrm{T} a_i > 0 > \min_{j \in \mathrm{supp}^-(x)} x^\mathrm{T} a_j$.

[Case 2: $x(N) = 0$] Putting $x' = x - \varepsilon \chi_i$ with a sufficiently small $\varepsilon > 0$, we have $x'(N) < 0$, $\mathrm{supp}^+(x') = \mathrm{supp}^+(x)$, and $\mathrm{supp}^-(x') = \mathrm{supp}^-(x)$. Hence, the argument in Case 1 implies

$$x^\mathrm{T} a_i - \varepsilon a_{ii} = (x')^\mathrm{T} a_i > \min_{j \in \mathrm{supp}^-(x)} (x')^\mathrm{T} a_j = \min_{j \in \mathrm{supp}^-(x)} (x^\mathrm{T} a_j - \varepsilon a_{ij}),$$

where $a_{ii} \geq a_{ij}$ by Lemma 6.9. This shows (ii). □

6.2.3 Proofs of Theorems 3.10 and 3.14 (ii)

Lemma 6.11 ($\mathrm{M}^\natural$-EXC) $\iff$ ($\mathrm{M}^\natural$-EXC_d).

Proof. ($\mathrm{M}^\natural$-EXC) for f can be rewritten as follows in terms of A (cf. (6.3)):

$$\begin{aligned} &\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x),\ \exists j \in \mathrm{supp}^-(x) \cup \{0\} : \\ &\qquad [x^\mathrm{T} a_j < x^\mathrm{T} a_i] \text{ or } [x^\mathrm{T} a_j = x^\mathrm{T} a_i \text{ and } a_{ii} + a_{jj} - 2a_{ij} \leq 0], \end{aligned} \tag{6.4}$$

where $a_0 = \mathbf{0}$ and $a_{i0} = a_{0i} = 0$ $(i \in N)$ by convention. Hence, the "$\Longrightarrow$" direction is obvious.

We then prove the converse. Let $x \in \mathbf{R}^n$ and $i \in \mathrm{supp}^+(x)$. For a sufficiently small $\varepsilon > 0$, define $\tilde{x} = x + 2\varepsilon\chi_S - 2k\varepsilon\chi_i$ with $S = \mathrm{supp}^-(x)$ and $k = |S|$. Then, we have $i \in \mathrm{supp}^+(\tilde{x}) = \mathrm{supp}^+(x)$ and $\mathrm{supp}^-(\tilde{x}) = \mathrm{supp}^-(x)$. By ($\mathrm{M}^\natural$-$\mathrm{EXC}_\mathrm{d}$) applied to $\tilde{x}$ and i, there exists $j \in \mathrm{supp}^-(\tilde{x}) \cup \{0\}$ such that

$$0 \geq \tilde{x}^\mathrm{T}(a_j - a_i) = x^\mathrm{T}(a_j - a_i) + (2\varepsilon\chi_S - 2k\varepsilon\chi_i)^\mathrm{T}(a_j - a_i). \tag{6.5}$$

Put $x' = 2\chi_S - \chi_j - (2k-1)\chi_i$. If $j \neq 0$, then ($\mathrm{M}^\natural$-EXC_d) applied to x' and j implies

$$(x')^\mathrm{T} a_j \geq \min\{0, (x')^\mathrm{T} a_i\} \geq (x')^\mathrm{T} a_i, \tag{6.6}$$

where the second inequality is by Lemma 6.9. Note that the inequality (6.6) also holds with $j = 0$ since $(x')^{\mathrm{T}} a_0 = 0$. Combining (6.5) and (6.6), we have

$$0 \geq x^{\mathrm{T}}(a_j - a_i) + \varepsilon(\chi_j - \chi_i)^{\mathrm{T}}(a_j - a_i) = x^{\mathrm{T}}(a_j - a_i) + \varepsilon(a_{ii} + a_{jj} - 2a_{ij}),$$

implying (6.4). □

The equivalence between (M-EXC) and (M-EXC$_{\mathrm{d}}$) can be shown similarly as Lemma 6.11, implying Theorem 3.14 (ii).

We denote by (M$^\natural$-PRJ$_{\mathrm{d}}$) the property (M$^\natural$-PRJ$_{\mathrm{d}}^+$) with the strict inequalities "$>$" replaced by inequalities "$\geq$". By (3.13) and Theorem 3.14 (ii), we have "$A \in \overline{\mathcal{M}}^\natural \iff \widehat{A} \in \overline{\mathcal{M}} \iff$ (M-EXC$_{\mathrm{d}}$) for $\widehat{A} \iff$ (M$^\natural$-PRJ$_{\mathrm{d}}$) for A," where $\widehat{A} = (\widehat{a}_{ij}) \in \mathcal{S}_{n+1}$ is given by (3.12). To conclude the proof of Theorem 3.10, it suffices to show "(M$^\natural$-PRJ$_{\mathrm{d}}$) $\iff$ (M$^\natural$-EXC$_{\mathrm{d}}$)." The "$\Longrightarrow$" direction is obvious. To show the converse, assume (M$^\natural$-EXC$_{\mathrm{d}}$) for A. For any $\beta > 0$, we define $A_\beta = A + \beta I$. Then, for $x \in \mathbf{R}^n$ and $i \in \mathrm{supp}^+(x)$ we have

$$\begin{aligned} x^{\mathrm{T}} a_i + \beta x_i \ > \ x^{\mathrm{T}} a_i &\geq \min\{0, \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j\} \\ &\geq \min\{0, \min_{j \in \mathrm{supp}^-(x)} (x^{\mathrm{T}} a_j + \beta x_j)\}, \end{aligned}$$

i.e., (M$^\natural$-EXC$_{\mathrm{d}}^+$) holds for A. From Lemma 6.10 follows (M$^\natural$-PRJ$_{\mathrm{d}}^+$) for A_β for any $\beta > 0$, implying (M$^\natural$-PRJ$_{\mathrm{d}}$) for A.

6.2.4 Proof of Theorems 3.11 and 3.15

We show the "only if" part of Theorem 3.11 only. The "if" part of Theorem 3.11 is easy, and Theorem 3.15 can be shown similarly.

Consider a quadratic M$^\natural$-convex function $f : \mathbf{R}^n \to \mathbf{R}$ given by (3.4) with $A \in \overline{\mathcal{M}}^\natural$. Since $\arg\min f$ is an M$^\natural$-convex linear subspace, there exists a subpartition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N such that

$$\arg\min f = \mathrm{span}(\bigcup_{k=1}^{m} \{\chi_j - \chi_i \mid i, j \in N_k\} \cup \{\chi_i \mid i \in N \setminus \bigcup_{k=1}^{m} N_k\})$$

by Proposition 2.2. Since $\ker A = \arg\min f$, we have

$$\begin{aligned} & a_h^{\mathrm{T}}(\chi_j - \chi_i) = a_{jh} - a_{ih} = 0 \quad (h \in N,\ i, j \in N_k,\ k \in \{1, 2, \ldots, m\}), \quad (6.7) \\ & a_h^{\mathrm{T}} \chi_i = a_{ih} = 0 \quad (h \in N,\ i \in N \setminus \bigcup_{k=1}^{m} N_k). \end{aligned}$$

We define a matrix $\widetilde{A} \in \mathcal{S}_m$ by $\tilde{a}_{kl} = a_{ij}$ $(i \in N_k,\ j \in N_l,\ k, l \in \{1, 2, \ldots, m\})$, which is well-defined by (6.7). Since $\widetilde{A}$ is a principal submatrix of A, $\widetilde{A}$ also belongs to $\overline{\mathcal{M}}^\natural$. Moreover, the equation $\ker A = \arg\min f$ implies that $\widetilde{A}$ is nonsingular. Hence, $\widetilde{A} \in \mathcal{M}^\natural$ holds.

6.2.5 *Proofs of Theorems 3.12 and 3.16*

We prove Theorem 3.16 only. Theorem 3.12 is immediate from Theorem 3.16.

We first show the "if" part. The function f is rewritten as

$$f(x) = \sum_{k=1}^{m} \frac{1}{2} x[N_k]^{\mathrm{T}}\, A[N_k]\, x[N_k] \qquad (x \in H).$$

Hence, the M-convexity of f follows immediately from $A[N_k] \in \overline{\mathcal{M}}$ $(k = 1, 2, \ldots, m)$.

We then prove the "only if" part. Suppose that f is M-convex. Then, $\mathrm{dom}\, f = H$ is an M-convex linear subspace, and therefore $H = \{x \in \mathbf{R}^n \mid x(N_k) = 0\ (k = 1, 2, \ldots, m)\}$ for some partition $\{N_k \mid k = 1, 2, \ldots, m\}$ of N by Proposition 2.2.

We then prove $A[N_k] \in \overline{\mathcal{M}}$ $(k = 1, 2, \ldots, m)$. Let $x, y \in H$ be any vectors with $x_h = y_h = 0$ $(h \in N \setminus N_k)$, and $i \in \mathrm{supp}^+(x - y) \cap N_k$. By the M-convexity of f, there exist some $j \in \mathrm{supp}^-(x - y)$ and $\alpha_0 > 0$ satisfying

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (\forall \alpha \in (0, \alpha_0]).$$

In particular, we have $x - \alpha(\chi_i - \chi_j) \in H$ and $y + \alpha(\chi_i - \chi_j) \in H$ for any $\alpha \in (0, \alpha_0]$, implying $j \in N_k$. This fact shows that the quadratic function $f_k : \mathbf{R}^{N_k} \to \mathbf{R} \cup \{+\infty\}$ given by

$$f_k(x') = \begin{cases} \frac{1}{2}(x')^{\mathrm{T}} A[N_k] x' & (x' \in \mathbf{R}^{N_k},\ x'(N_k) = 0), \\ +\infty & (\text{otherwise}) \end{cases}$$

is M-convex, implying $A[N_k] \in \overline{\mathcal{M}}$.

We next show (3.8). Put $x = (\chi_i - \chi_{i'}) + \beta(\chi_j - \chi_{j'})$ with any real number β, and $y = \mathbf{0}$. Then, (M-EXC) applied to $x, y \in H$ and $i \in \mathrm{supp}^+(x - y)$ implies that there exists some $\alpha > 0$ such that

$$0 \geq \alpha(2a_{ii'} - a_{ii} - a_{i'i'}) + \alpha\beta(a_{i'j} - a_{i'j'} - a_{ij} + a_{ij'}) + \alpha^2(a_{ii} + a_{i'i'} - 2a_{ii'}).$$

Since β can be chosen arbitrarily, we have $a_{i'j} - a_{i'j'} - a_{ij} + a_{ij'} = 0$.

Acknowledgements

The authors thank Yoichiro Takahashi for discussion and bibliographic information, and anonymous referees for their valuable comments on the manuscript. This work is supported by Grant-in-Aid of the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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