Level Set Characterization of M-convex Functions

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Abstract

This note shows that level sets of a function characterize an M-convex function introduced by Murota.

Keywords: matroid, base polyhedron, convex function, level set.

1 Result

The concept of M-convex function, introduced by Murota [4, 5, 6], is an extension of valuated matroid due to Dress and Wenzel [1, 2] as well as a quantitative generalization of (the integral points of) the base polyhedron of an integral submodular system [3]. Let V be a finite set. It is known as a folklore that (the set of integral vectors in) an integral base polyhedron $B \subseteq \mathbf{Z}^V$ can be characterized as a nonempty set satisfying the property:

(B-EXC) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$x - \chi_u + \chi_v \in B, \quad y + \chi_u - \chi_v \in B,$$

where $\mathrm{supp}^+(x-y) = \{w \in V \mid x(w) > y(w)\}$, $\mathrm{supp}^-(x-y) = \{w \in V \mid x(w) < y(w)\}$, and $\chi_w \in \{0,1\}^V$ is the characteristic vector of $w \in V$. In contrast, an M-convex function is defined to be a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ satisfying the next property:

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where $\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}$. An example of M-convex function is a separable-convex function defined over an integral base polyhedron, and more general M-convex functions arise from the minimum cost flow problems with separable-convex cost functions. M-convex functions enjoy various nice properties which are sufficient for us to regard M-convexity as "convexity" in combinatorial optimization. Minimization of an M-convex function can be done in weakly-polynomial time [7].

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The aim of this note is to provide a characterization of M-convex functions by level sets. Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. For $\alpha \in \mathbf{R}$, we denote by $\mathrm{L}(f, \alpha)$ the level set $\{x \in \mathbf{Z}^V \mid f(x) \leq \alpha\}$. For $p \in \mathbf{R}^V$, define the function $f[p] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by $f[p](x) = f(x) + \sum_{w \in V} p(w)x(w)$ $(x \in \mathbf{Z}^V)$. If f is an M-convex function with bounded $\mathrm{dom}\, f$, then for any $p \in \mathbf{R}^V$ the level set $\mathrm{L}(f[p], \min f[p])$, i.e., the set of minimizers of $f[p]$, is an integral base polyhedron. In fact, the converse also holds true:

Theorem 1 ([5, Theorem 4.4]) : *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be such that* $\mathrm{dom}\, f$ *is a bounded integral base polyhedron. Then, f is M-convex if and only if for any $p \in \mathbf{R}^V$, $\{x \in \mathbf{Z}^V \mid f[p](x) \leq f[p](y)\ (\forall y \in \mathbf{Z}^V)\}$ is an integral base polyhedron.*

(see [6, Theorem 4.10] for the general case of unbounded effective domain). In general, however, a level set of an M-convex function is not necessarily an integral base polyhedron:

Example. Define the function $f : \mathbf{Z}^4 \to \mathbf{R} \cup \{+\infty\}$ as

$$\mathrm{dom}\, f = \{(1,1,0,0), (0,0,1,1), (1,0,1,0), (0,1,0,1)\},$$
$$f(1,1,0,0) = 0,\ f(0,0,1,1) = 2,\ f(1,0,1,0) = 1,\ f(0,1,0,1) = 1.$$

While f is M-convex, the level set $\mathrm{L}(f, 1)$ does not satisfy (B-EXC). □

In this note, we show that a function is M-convex if and only if any level set of the function satisfies either of the following exchange properties, where $S \subseteq \mathbf{Z}^V$:

(EXC) $\forall x, y \in S$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that
either $x - \chi_u + \chi_v \in S$ or $y + \chi_u - \chi_v \in S$,

(EXC$_\mathbf{w}$) $\forall x, y \in S$, $x \neq y$, $\exists u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that
either $x - \chi_u + \chi_v \in S$ or $y + \chi_u - \chi_v \in S$,

Theorem 2. *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with* $\mathrm{dom}\, f \neq \emptyset$*, the following conditions are equivalent:*
(i) f is M-convex.
(ii) $\forall p \in \mathbf{R}^V$, $\forall \alpha \in \mathbf{R}$, $\mathrm{L}(f[p], \alpha)$ satisfies (EXC).
(iii) $\forall p \in \mathbf{R}^V$, $\forall \alpha \in \mathbf{R}$, $\mathrm{L}(f[p], \alpha)$ satisfies (EXC$_w$).

2 Proof of Theorem 2

We firstly show some useful properties. M-convexity can be characterized by local M-convexity:

(M-EXC$_\mathbf{loc}$) $\forall x, y \in \mathrm{dom}\, f$, $||x - y|| = 4$, $\exists u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where $||z|| = \sum_{w \in V} |z(w)|$ $(z \in \mathbf{Z}^V)$. Compare the next theorem with [5, Theorem 3.1], where $\mathrm{dom}\, f$ is assumed to satisfy (B-EXC) instead of (EXC$_\mathrm{w}$).

Theorem 3. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, and suppose that* $\mathrm{dom}\, f$ *is nonempty and satisfies (EXC$_w$). Then, (M-EXC) $\Longleftrightarrow$ (M-EXC$_{loc}$).*

Proof. The proof is almost the same as that of [5, Theorem 3.1]. See Appendix for details. ∎

Lemma 4. *For $S \subseteq \mathbf{Z}^V$, (EXC) $\Longrightarrow$ (EXC$_w$).*

Lemma 5. *Suppose $S \subseteq \mathbf{Z}^V$ satisfies (EXC$_w$). Then, $x(V) = y(V)$ for any $x, y \in \mathrm{dom}\, f$.*

We now prove Theorem 2.

(i) $\Longrightarrow$ (ii): Let $p \in \mathbf{R}^V$, $\alpha \in \mathbf{R}$, and x, y be distinct vectors in $\mathrm{L}(f[p], \alpha)$. By (M-EXC) for x, y and $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that $f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)$. Hence,

$$2\alpha \geq f[p](x) + f[p](y) \geq f[p](x - \chi_u + \chi_v) + f[p](y + \chi_u - \chi_v),$$

which implies either $f[p](x - \chi_u + \chi_v) \leq \alpha$ or $f[p](y + \chi_u - \chi_v) \leq \alpha$.

(ii) $\Longrightarrow$ (iii): Obvious from Lemma 4.

(iii) $\Longrightarrow$ (i): Suppose $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ satisfies (iii).

Claim 1: $\mathrm{dom}\, f$ satisfies (EXC$_\mathrm{w}$).

Let $x, y \in \mathrm{dom}\, f$ and $\alpha = \max\{f(x), f(y)\}$. Then (EXC$_\mathrm{w}$) for $x, y \in \mathrm{L}(f, \alpha)$ implies that either $x - \chi_u + \chi_v \in \mathrm{L}(f, \alpha) \subseteq \mathrm{dom}\, f$ or $y + \chi_u - \chi_v \in \mathrm{L}(f, \alpha) \subseteq \mathrm{dom}\, f$ holds for some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$. [End of Claim 1]

By Theorem 3, it suffices to show (M-EXC$_\mathrm{loc}$) for f. Let $x, y \in \mathrm{dom}\, f$ with $||x - y|| = 4$. We have $y = x - \chi_{u_0} - \chi_{u_1} + \chi_{v_0} + \chi_{v_1}$ for some $u_0, u_1, v_0, v_1 \in V$ with $\{u_0, u_1\} \cap \{v_0, v_1\} = \emptyset$. For $i, j \in \{0, 1\}$, denote $z_{ij} = x \wedge y + \chi_{u_i} + \chi_{v_j}$, where $x \wedge y \in \mathbf{R}^V$ is the vector with $(x \wedge y)(w) = \min\{x(w), y(w)\}$ $(w \in V)$. Then, (M-EXC$_\mathrm{loc}$) for x, y is rewritten as

$$f(x) + f(y) \geq \min\{f(z_{00}) + f(z_{11}), f(z_{01}) + f(z_{10})\}. \tag{1}$$

Claim 2: If $f[p](x) = f[p](y) \geq f[p](z_{i,j}) = f[p](z_{1-i,1-j})$ holds for some $p \in \mathbf{R}^V$ and $i, j \in \{0, 1\}$, then we have the inequality (1). [End of Claim 2]

[Case 1: $u_0 = u_1$, $v_0 = v_1$] In this case, $z_{00} = z_{10} = z_{01} = z_{11}$. Let $p \in \mathbf{R}^V$ be the vector with

$$p(u_0) = \{f(y) - f(x)\}/2, \quad p(w) = 0 \ (\forall w \in V - u_0).$$

Then, $f[p](x) = f[p](y)$. The property (EXC$_\mathrm{w}$) for $x, y \in \mathrm{L}(f[p], f[p](x))$ implies $f[p](z_{00}) \leq f[p](x)$. Thus, (1) follows by Claim 2.

[Case 2: $u_0 = u_1$, $v_0 \neq v_1$] In this case, $z_{00} = z_{10}$ and $z_{11} = z_{01}$ hold. We first show that $z_{00}, z_{11} \in \mathrm{dom}\, f$. By (EXC$_\mathrm{w}$) for $\mathrm{dom}\, f$, w.l.o.g. assume that $z_{00} \in \mathrm{dom}\, f$. Define $q \in \mathbf{R}^V$ by

$$q(v_0) = f(x) - f(z_{00}) + 1, \ q(v_1) = f(x) - f(y) - p(v_0), \ q(w) = 0 \ (\forall w \in V - \{v_0, v_1\}).$$

Then, $f[q](x) = f[q](y)$, $f[q](z_{00}) > f[q](x)$. Hence, we have $z_{11} \in \mathrm{L}(f[q], f[q](x)) \subseteq \mathrm{dom}\, f$ by applying ($\mathrm{EXC_w}$) to $x, y \in \mathrm{L}(f[q], f[q](x))$.

Let $p \in \mathbf{R}^V$ be the vector with

$$p(v_0) = f(z_{11}) - f(z_{00}), \quad p(u_0) = (1/2)\{f(y) - f(x) + p(v_0)\},$$
$$p(w) = 0 \ (\forall w \in V - \{u_0, v_0\}).$$

Then, it holds $f[p](x) = f[p](y)$ and $f[p](z_{00}) = f[p](z_{11})$. Furthermore, we have $f[p](z_{00}) = f[p](z_{11}) \leq f[p](x)$ by ($\mathrm{EXC_w}$) for $x, y \in \mathrm{L}(f[p], f[p](x))$, which implies (1) by Claim 2.

[Case 3: $u_0 \neq u_1$, $v_0 = v_1$] Similar to Case 2 and omitted.

[Case 4: $u_0 \neq u_1$, $v_0 \neq v_1$] We first show by contradiction that either $z_{00}, z_{11} \in \mathrm{dom}\, f$ or $z_{01}, z_{10} \in \mathrm{dom}\, f$ holds. Assume w.l.o.g. that $z_{11}, z_{10} \notin \mathrm{dom}\, f$. Let $q \in \mathbf{R}^V$ satisfy

$$q(u_0) = f(y) - f(x) - q(u_1), \quad q(u_1) < \min\{f(z_{00}), f(z_{01})\} - f(x),$$
$$q(w) = 0 \ (\forall w \in V - \{u_0, u_1\}).$$

Then, we have $f[q](x) = f[q](y)$, $f[q](z_{00}) > f[q](x)$, and $f[q](z_{01}) > f[q](x)$. Hence, none of $z_{00}, z_{01}, z_{10}, z_{11}$ is contained in $\mathrm{L}(f[q], f[q](x))$, which contradicts ($\mathrm{EXC_w}$) for $x, y \in \mathrm{L}(f[q], f[q](x))$.

We w.l.o.g. assume $z_{00}, z_{11} \in \mathrm{dom}\, f$. Let $p \in \mathbf{R}^V$ be the vector satisfying

$$p(u_0) = f(y) - f(x) + p(v_0) + p(v_1), \quad p(v_0) = (1/2)\{f(x) + f(z_{11}) - f(y) - f(z_{00})\},$$
$$p(w) = 0 \ (\forall w \in V - \{u_0, v_0, v_1\}).$$

The value $p(v_1)$ is specified later. Then, we have $f[p](x) = f[p](y)$, $f[p](z_{00}) = f[p](z_{11})$.

[Case 4-1: $z_{10} \notin \mathrm{dom}\, f$] Set $p(v_1)$ to any number with $p(v_1) > f(x) - f(z_{01})$. Since $f[p](z_{01}) > f[p](x)$, it holds $z_{01}, z_{10} \notin \mathrm{L}(f[p], f[p](x))$. Therefore, we have $f[p](z_{00}) = f[p](z_{11}) \leq f[p](x)$ by ($\mathrm{EXC_w}$) for $x, y \in \mathrm{L}(f[p], f[p](x))$. Hence, (1) holds by Claim 2.

[Case 4-2: $z_{01} \notin \mathrm{dom}\, f$] Similar to Case 4-1 and omitted.

[Case 4-3: $z_{01}, z_{10} \in \mathrm{dom}\, f$] Set $p(v_1) = (1/2)(f(x) + f(z_{10}) - f(y) - f(z_{01}))$. Then, $f[p](z_{01}) = f[p](z_{10})$. Applying ($\mathrm{EXC_w}$) to $x, y \in \mathrm{L}(f[p], f[p](x))$, we have either $f[p](z_{00}) = f[p](z_{11}) \leq f[p](x)$ or $f[p](z_{01}) = f[p](z_{10}) \leq f[p](x)$. Therefore, Claim 2 implies (1).

Remark 1. Theorem 2 can be specialized to integral base polyhedra as follows:

Corollary 6. *For $B \subseteq \mathbf{Z}^V$ with $B \neq \emptyset$, the following conditions are equivalent:*
(i) B is an integral base polyhedron.
(ii) $\forall p \in \{0, \pm 1\}^V$, $\forall \alpha \in \mathbf{Z}$, the set $\{x \in B \mid \sum_{w \in V} p(w)x(w) \leq \alpha\}$ satisfies (EXC).
(iii) $\forall p \in \{0, \pm 1\}^V$, $\forall \alpha \in \mathbf{Z}$, the set $\{x \in B \mid \sum_{w \in V} p(w)x(w) \leq \alpha\}$ satisfies (EXC_w).

Note that the restrictions $p \in \{0, \pm 1\}^V$ and $\alpha \in \mathbf{Z}$ follow from the proof of Theorem 2, and not from the statement itself. $\square$

Remark 2. ($\mathrm{EXC_w}$) is equivalent to a seemingly stronger exchange property:

($\mathbf{EXC_+}$) $\forall x, y \in S$, $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in S$.

Theorem 7. *For any $S \subseteq \mathbf{Z}^V$, (EXC_w) $\iff$ (EXC_+).*

This equivalence is easy to prove. $\square$

Remark 3. While (B-EXC) $\Longrightarrow$ (EXC) $\Longrightarrow$ ($\mathrm{EXC_w}$) holds true, the reverse implications do not holds necessarily. Let $V = \{1,2,3,4,5,6\}$ and

$$\mathcal{S}_1 = \{\{1,2\},\{3,4\},\{1,3\}\}, \quad \mathcal{S}_2 = \{\{1,2,3\},\{1,2,4\},\{1,4,5\},\{4,5,6\}\}.$$

Denote by S_1, S_2 the sets of characteristic vectors of subsets in $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Then, S_1 satisfies (EXC) but not (B-EXC), and S_2 satisfies ($\mathrm{EXC_w}$) but not (EXC). $\square$

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Appendix: Proof of Theorem 3

We prove ($\mathrm{M\text{-}EXC_{loc}}$) $\Longrightarrow$ (M-EXC) only. Firstly note that $x(V) = y(V)$ ($\forall x, y \in \mathrm{dom}\, f$) by Lemma 5. Define

$$\mathcal{D} = \{(x,y) \mid x, y \in \mathrm{dom}\, f,\ \exists u_* \in \mathrm{supp}^+(x-y),\ \forall v \in \mathrm{supp}^-(x-y) : \\ f(x) + f(y) < f(x - \chi_{u_*} + \chi_v) + f(y + \chi_{u_*} - \chi_v)\}.$$

To the contrary assume $\mathcal{D} \neq \emptyset$. Let $(x,y) \in \mathcal{D}$ minimize the value $||x-y||$, and $u_* \in \mathrm{supp}^+(x-y)$ satisfy the condition for (x,y) to be in $\mathcal{D}$. We have $||x-y|| \geq 6$ by ($\mathrm{M\text{-}EXC_{loc}}$). Define $p \in \mathbf{R}^V$ by

$$p(w) = \begin{cases} f(x) - f(x - \chi_{u_*} + \chi_v) & (v \in \mathrm{supp}^-(x-y),\ x - \chi_{u_*} + \chi_v \in \mathrm{dom}\, f), \\ f(y + \chi_{u_*} - \chi_v) - f(y) - \varepsilon & (v \in \mathrm{supp}^-(x-y),\ x - \chi_{u_*} + \chi_v \notin \mathrm{dom}\, f, \\ & \qquad y + \chi_{u_*} - \chi_v \in \mathrm{dom}\, f), \\ 0 & (\text{otherwise}) \end{cases}$$

with some $\varepsilon > 0$.

Claim 1:

$$f[p](x - \chi_{u_*} + \chi_v) = f[p](x) \quad (v \in \mathrm{supp}^-(x-y),\ x - \chi_{u_*} + \chi_v \in \mathrm{dom}\, f), \tag{2}$$

$$f[p](y + \chi_{u_*} - \chi_v) > f[p](y) \quad (v \in \mathrm{supp}^-(x-y)). \tag{3}$$

The equality (2) immediately follows from the definition of p. For $v \in \mathrm{supp}^-(x-y)$, if $x - \chi_{u_*} + \chi_v \in \mathrm{dom}\, f$ then

$$f[p](y+\chi_{u_*}-\chi_v) - f[p](y) = f(y+\chi_{u_*}-\chi_v) - f(y) - f(x) + f(x-\chi_{u_*}+\chi_v) \ > \ 0.$$

Otherwise the value $f[p](y+\chi_{u_*}-\chi_v) - f[p](y)$ is equal to ε or $+\infty$ according to whether $y+\chi_{u_*}-\chi_v \in \mathrm{dom}\, f$ or not. [End of Claim 1]

Claim 2: There exists $u_0 \in \mathrm{supp}^+(x-y)$ such that $y+\chi_{u_0}-\chi_v \in \mathrm{dom}\, f$ for some $v \in \mathrm{supp}^-(x-y)$. Moreover, if $x(u_*)-y(u_*)=1$ then we can assume $u_0 \neq u_*$.

By Theorem 7, $\mathrm{dom}\, f$ satisfies (EXC$_+$). Applying (EXC$_+$) to y and x, there exist $u_1 \in \mathrm{supp}^+(x-y)$ and $v_1 \in \mathrm{supp}^-(x-y)$ with $y_1 = y+\chi_{u_1}-\chi_{v_1} \in \mathrm{dom}\, f$. Since $||x-y_1|| \geq 4$, we can again apply (EXC$_+$) to y_1, x and obtain $u_2 \in \mathrm{supp}^+(x-y_1) \subseteq \mathrm{supp}^+(x-y)$, $v_2 \in \mathrm{supp}^-(x-y_1) \subseteq \mathrm{supp}^-(x-y)$ with $y_2 = y_1+\chi_{u_2}-\chi_{v_2} \in \mathrm{dom}\, f$. Note that if $x(u_*)-y(u_*)=1$ then either u_1 or u_2 is different from u_*. Applying (M-EXC$_{\mathrm{loc}}$) to y and y_2, we have

$$\begin{aligned} f(y)+f(y_2) \geq \min\{ & f(y+\chi_{u_1}-\chi_{v_1}) + f(y+\chi_{u_2}-\chi_{v_2}), \\ & f(y+\chi_{u_1}-\chi_{v_2}) + f(y+\chi_{u_2}-\chi_{v_1})\}, \end{aligned}$$

which yields either $y+\chi_{u_1}-\chi_{v_1},\ y+\chi_{u_2}-\chi_{v_2} \in \mathrm{dom}\, f$ or $y+\chi_{u_1}-\chi_{v_2},\ y+\chi_{u_2}-\chi_{v_1} \in \mathrm{dom}\, f$. Thus, we can choose u_0 from $\{u_1,u_2\}-u_*$ if $x(u_*)-y(u_*)=1$, and from $\{u_1,u_2\}$ if $x(u_*)-y(u_*)>1$. [End of Claim 2]

Define $v_0 \in V$ as the element in $\mathrm{supp}^-(x-y)$ satisfying

$$f[p](y+\chi_{u_0}-\chi_{v_0}) \leq f[p](y+\chi_{u_0}-\chi_v) \quad (\forall v \in \mathrm{supp}^-(x-y)). \tag{4}$$

Claim 3: $(x,y') \in \mathcal{D}$ holds for $y' = y+\chi_{u_0}-\chi_{v_0}$.

By Claim 2, $y' \in \mathrm{dom}\, f$ and $u_* \in \mathrm{supp}^+(x-y')$. It suffices to show that

$$f[p](x) + f[p](y') < f[p](x-\chi_{u_*}+\chi_v) + f[p](y'+\chi_{u_*}-\chi_v) \tag{5}$$

for $v \in \mathrm{supp}^-(x-y')$. We may assume that $x-\chi_{u_*}+\chi_v,\ y'+\chi_{u_*}-\chi_v \in \mathrm{dom}\, f$, otherwise (5) holds obviously. Then, we have the inequality

$$\begin{aligned} & f[p](y'+\chi_{u_*}-\chi_v) - f[p](y') \\ = \ & \{f[p](y+\chi_{u_0}+\chi_{u_*}-\chi_{v_0}-\chi_v) + f[p](y)\} - f[p](y+\chi_{u_0}-\chi_{v_0}) - f[p](y) \\ \geq \ & \min\{f[p](y+\chi_{u_0}-\chi_{v_0}) + f[p](y+\chi_{u_*}-\chi_v),\ f[p](y+\chi_{u_0}-\chi_v) + f[p](y+\chi_{u_*}-\chi_{v_0})\} \\ & \qquad -f[p](y+\chi_{u_0}-\chi_{v_0}) - f[p](y) \quad (\text{by (M-EXC}_{\mathrm{loc}})) \\ \geq \ & \min\{f[p](y+\chi_{u_*}-\chi_v) - f[p](y), f[p](y+\chi_{u_*}-\chi_{v_0}) - f[p](y)\} \quad (\text{by (4)}) \\ > \ & 0 \quad (\text{by (3)}), \end{aligned}$$

which, together with (2), implies (5). [End of Claim 3]

Since $||x-y'|| = ||x-y||-2$, Claim 3 contradicts the choice of (x,y).

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