Polynomial-Time Approximation Schemes for Maximizing Gross Substitutes Utility under Budget Constraints

Akiyoshi Shioura

GSIS, Tohoku University, Sendai 980-8579, Japan, shioura@dais.is.tohoku.ac.jp

Abstract. We consider the maximization of a gross substitutes utility function under budget constraints. This problem naturally arises in applications such as exchange economies in mathematical economics and combinatorial auctions in (algorithmic) game theory. We show that this problem admits a polynomial-time approximation scheme (PTAS). More generally, we present a PTAS for maximizing a discrete concave function called an $\mathrm{M}^\natural$-concave function under budget constraints. Our PTAS is based on rounding an optimal solution of a continuous relaxation problem, which is shown to be solvable in polynomial time by the ellipsoid method. We also consider the maximization of the sum of two $\mathrm{M}^\natural$-concave functions under a single budget constraint. This problem is a generalization of the budgeted max-weight matroid intersection problem to the one with a nonlinear objective function. We show that this problem also admits a PTAS.

1 Introduction

We consider the problem of maximizing a nonlinear utility function under a constant number of budget (or knapsack) constraints, which is formulated as

$$\text{Maximize } f(X) \text{ subject to } X \in 2^N,\ c_i(X) \leq B_i \ (1 \leq i \leq k), \tag{1}$$

where N is a set of n items, $f : 2^N \to \mathbb{R}$ is a nonlinear utility function[1] of a consumer (or buyer) with $f(\emptyset) = 0$, k is a constant positive integer, and $c_i \in \mathbb{R}^N_+$, $B_i \in \mathbb{R}_+$ $(i = 1, 2, \ldots, k)$. For a vector $a \in \mathbb{R}^N$ and a set $X \subseteq N$, we denote $a(X) = \sum_{v \in X} a(v)$. The problem (1) is a natural generalization of budgeted combinatorial optimization problems ([20, 21, 38], etc.), and naturally arises in applications such as exchange economies with indivisible objects in mathematical economics ([18, 19], etc.) and combinatorial auctions in (algorithmic) game theory ([4, 7, 22], etc.).

The problem (1) with a submodular objective function f is extensively discussed in the literature of combinatorial optimization, and constant-factor approximation algorithms have been proposed. Wolsey [39] considered the problem

[1] Monotonicity of f is not assumed throughout this paper, although utility functions are often assumed to be monotone.

(1) with a monotone submodular f and $k = 1$, and proposed the first constant-factor approximation algorithm with the ratio $1 - e^{-\beta} \simeq 0.35$, where β satisfies $e^{\beta} = 2 - \beta$. Later, Sviridenko [38] improved the approximation ratio to $1 - 1/e$, which is the best possible under the assumption that P $\neq$ NP [10]. For the case of a monotone submodular f and a general constant $k \geq 1$, Kulik et al. [20] proposed a $(1 - 1/e)$-approximation algorithm by using the approach of Calinescu et al. [5] for the submodular function maximization under a matroid constraint. For a non-monotone submodular f and a general constant $k \geq 1$, a $(1/5 - \varepsilon)$-approximation local-search algorithm was given by Lee et al. [21].

Submodularity for set functions is known to be equivalent to the concept of decreasing marginal utility in mathematical economics. In this paper, we focus on a more specific subclass of decreasing marginal utilities, called *gross substitutes utilities*, and show that the problem (1) admits a polynomial-time approximation scheme (PTAS) if f is a gross substitutes utility.

A *gross substitutes utility* (*GS utility*, for short) function is defined as a function $f : 2^N \to \mathbb{R}$ satisfying the following condition:

$$\forall p, q \in \mathbb{R}^N \text{ with } p \leq q,\ \forall X \in \arg\max_{U \subseteq N}\{f(U) - p(U)\},$$
$$\exists Y \in \arg\max_{U \subseteq N}\{f(U) - q(U)\} \text{ s.t. } \{v \in X \mid p(v){=}q(v)\} \subseteq Y,$$

where p and q represent price vectors. This condition means that a consumer still wants to get items that do not change in price after the prices on other items increase. The concept of GS utility is introduced in Kelso and Crawford [19], where the existence of a Walrasian (or competitive) equilibrium is shown in a fairly general two-sided matching model. Since then, this concept plays a central role in mathematical economics and in auction theory, and is widely used in various models such as matching, housing, and labor market (see, e.g., [1, 3, 4, 7, 15, 18, 22]).

Various characterizations of gross substitutes utilities are given in the literature of mathematical economics [1, 15, 18]. Among them, Fujishige and Yang [15] revealed the relationship between GS utilities and discrete concave functions called *M$^\natural$-concave functions*, which is a function on matroid independent sets. It is known that a family $\mathcal{F} \subseteq 2^N$ of matroid independent sets satisfies the following property [30]:

(B$^\natural$-EXC) $\forall X, Y \in \mathcal{F}$, $\forall u \in X \setminus Y$, at least one of (i) $X - u, Y + u \in \mathcal{F}$, and (ii) $\exists v \in Y \setminus X$: $X - u + v$, $Y + u - v \in \mathcal{F}$, holds,

where $X - u + v$ is a short-hand notation for $X \setminus \{u\} \cup \{v\}$. We consider a function $f : \mathcal{F} \to \mathbb{R}$ defined on matroid independent sets $\mathcal{F}$. A function f is said to be *M$^\natural$-concave* [30] (read "M-natural-concave") if it satisfies the following:[2]

(M$^\natural$-EXC) $\forall X, Y \in \mathcal{F}$, $\forall u \in X \setminus Y$, at least one of (i) $X - u, Y + u \in \mathcal{F}$ and $f(X) + f(Y) \leq f(X - u) + f(Y + u)$, and (ii) $\exists v \in Y \setminus X$: $X{-}u{+}v$, $Y{+}u{-}v \in \mathcal{F}$ and $f(X) + f(Y) \leq f(X{-}u{+}v) + f(Y{+}u{-}v)$, holds.

[2] The concept of M$^\natural$-concavity is originally introduced for functions defined on generalized (poly)matroids (see [30]). In this paper we mainly consider a restricted class of M$^\natural$-concave functions.

The concept of M$^\natural$-concave function is introduced by Murota and Shioura [30] as a class of discrete concave functions (independently of gross substitutes utilities). It is an extension of the concept of M-concave function introduced by Murota [25, 27]. The concepts of M$^\natural$-concavity/M-concavity play primary roles in the theory of discrete convex analysis [28], which provides a framework for well-solved nonlinear discrete optimization problems.

It is shown by Fujishige and Yang [15] that gross substitutes utilities constitute a subclass of M$^\natural$-concave functions.

Theorem 1.1. *A function $f : 2^N \to \mathbb{R}$ defined on 2^N is a gross substitutes utility if and only if f is an M$^\natural$-concave function.*

This result initiated a strong interaction between discrete convex analysis and mathematical economics; the results obtained in discrete convex analysis are used in mathematical economics ([3, 22], etc.), while mathematical economics provides interesting applications in discrete convex analysis ([32, 33], etc.).

In this paper, we mainly consider the *k-budgeted M$^\natural$-concave maximization problem*:

$$\textbf{(}k\textbf{BM}^\natural\textbf{M)}\ \text{Maximize } f(X) \text{ subject to } X \in \mathcal{F},\ c_i(X) \le B_i\ (1 \le i \le k),$$

which is (slightly) more general than the problem (1) with a gross-substitutes utility. Here, $f : \mathcal{F} \to \mathbb{R}$ is an M$^\natural$-concave function with $f(\emptyset) = 0$ defined on matroid independent sets $\mathcal{F}$, and k, c_i, and B_i are as in (1). We assume that the objective function f is given by a constant-time oracle which, given a subset $X \in 2^N$, checks if $X \in \mathcal{F}$ or not, and if $X \in \mathcal{F}$ then returns the value $f(X)$. The class of M$^\natural$-concave functions includes, as its subclass, linear functions on matroid independent sets. Hence, the problem (kBM$^\natural$M) is a nonlinear generalization of the max-weight matroid independent set problem with budget constraints, for which Grandoni and Zenklusen [16] has recently proposed a conceptually simple, deterministic PTAS using the polyhedral structure of matroids.

Our Main Result In this paper, we propose a PTAS for (kBM$^\natural$M) by extending the approach of Grandoni and Zenklusen [16]. We show the following property, where OPT denotes the optimal value of (kBM$^\natural$M).

Theorem 1.2. *A feasible solution $\tilde{X} \in 2^N$ of (kBM$^\natural$M) satisfying $f(\tilde{X}) \ge \text{OPT} - 2k \max_{v \in N} f(\{v\})$ can be computed deterministically in polynomial time.*

The algorithm used in Theorem 1.2 can be converted into a PTAS by using a standard technique called *partial enumeration*, which reduces the original problem to a family of problems with "small" elements, which is done by guessing a constant number of "large" elements contained in an optimal solution (see §A.1 in Appendix; see also [2, 16, 20, 34]). Hence, we obtain the following:

Theorem 1.3. *For every fixed $\varepsilon > 0$, a $(1-\varepsilon)$-approximate solution of (kBM$^\natural$M) can be computed deterministically in polynomial time.*

To prove Theorem 1.2, we use the following algorithm, which is a natural extension of the one in [16]:

STEP 1: Construct a continuous relaxation problem (CR) of (kBM$^\natural$M).
STEP 2: Compute a vertex optimal solution $\hat{x} \in [0,1]^N$ of (CR).
STEP 3: Round down the non-integral components of the optimal solution $\hat{x}$.

In [16], LP relaxation is used as a continuous relaxation, and it is shown that a vertex optimal solution (i.e., an optimal solution which is a vertex of the feasible region) of the resulting LP is nearly integral. Since the LP relaxation problem can be solved in polynomial time by the ellipsoid method, rounding down a vertex optimal solution yields a near-optimal solution of the budgeted max-weight matroid independent set problem.

These techniques in [16], however, cannot be applied directly since the objective function in (kBM$^\natural$M) is nonlinear. In particular, our continuous relaxation problem (CR) is a nonlinear programming problem formulated as

$$\textbf{(CR)} \quad \text{Maximize } \overline{f}(x) \quad \text{subject to } x \in \overline{\mathcal{F}},\ c_i^\top x \le B_i\ (1 \le i \le k). \qquad (2)$$

Here, $\overline{\mathcal{F}}$ is a matroid polytope of the matroid $(N, \mathcal{F})$ and $\overline{f}$ is a concave closure of the function f (see §3 for definitions).

To extend the approach in [16], we firstly modify the definition of vertex optimal solution appropriately since there may be no optimal solution which is a vertex of the feasible region if the objective function is nonlinear. Under the new definition, we show that a vertex optimal solution of (CR) is nearly integral by using the polyhedral structure of M$^\natural$-concave functions.

We then show that if f is an M$^\natural$-concave function, then (CR) can be solved (almost) optimally in polynomial time by the ellipsoid method [17]. Note that the function $\overline{f}$ is given implicitly, and the evaluation of the function value is still a nontrivial task; even if f is a monotone submodular function, the evaluation of $\overline{f}(x)$ is NP-hard [5]. To solve (CR) we use the following new algorithmic property concerning the concave closure of M$^\natural$-concave functions, which is proven by making full use of conjugacy results of M$^\natural$-concave functions in the theory of discrete convex analysis.

Lemma 1.1. *Let $x \in \overline{\mathcal{F}}$.*
(i) *For every $\delta > 0$, we can compute in polynomial time $p \in \mathbb{Q}^N$ and $\beta \in \mathbb{Q}$ satisfying $\overline{f}(y) - \overline{f}(x) \le p^\top (y - x) + \delta$ ($\forall y \in \overline{\mathcal{F}}$) and $\overline{f}(x) \le \beta \le \overline{f}(x) + \delta$.*
(ii) *If f is an integer-valued function, then we can compute in polynomial time $p \in \mathbb{Q}^N$ with $\overline{f}(y) - \overline{f}(x) \le p^\top (y - x)$ ($\forall y \in \overline{\mathcal{F}}$) and the value $\overline{f}(x)$.*

Our Second Result We also consider another type of budgeted optimization problem, which we call the *budgeted M$^\natural$-concave intersection problem*:

$$\textbf{(1BM}^\natural\textbf{I)} \quad \text{Maximize } f_1(X) + f_2(X) \quad \text{subject to } X \in \mathcal{F}_1 \cap \mathcal{F}_2,\ c(X) \le B,$$

where $f_j : \mathcal{F}_j \to \mathbb{R}$ $(j = 1, 2)$ are M$^\natural$-concave functions with $f_j(\emptyset) = 0$ defined on matroid independent sets $\mathcal{F}_j$, $c \in \mathbb{R}^N_+$ and $B \in \mathbb{R}_+$. This is a nonlinear generalization of the budgeted max-weight matroid intersection problem. Indeed, if each f_j is a linear function on matroid independent sets $\mathcal{F}_j$, then the problem (1BM$^\natural$I) is nothing but the budgeted max-weight matroid intersection problem, for which Berger et al. [2] proposed a PTAS using Lagrangian relaxation and a novel patching operation. We show that the approach can be extended to (1BM$^\natural$I).

Theorem 1.4. *For every fixed $\varepsilon > 0$, a $(1-\varepsilon)$-approximate solution of* (1BM$^\natural$I) *can be computed deterministically in polynomial time.*

The following is the key property to prove Theorem 1.4, where OPT denotes the optimal value of (1BM$^\natural$I). We may assume that $\{v\} \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $f_1(\{v\}) + f_2(\{v\}) > 0$ hold for all $v \in N$.

Theorem 1.5. *For* (1BM$^\natural$I)*, there exists a polynomial-time algorithm which computes a set $\tilde{X} \in \mathcal{F}_1 \cap \mathcal{F}_2$ satisfying $f_1(\tilde{X}) + f_2(\tilde{X}) \geq \text{OPT} - 2 \cdot \max_{v \in N}\{f_1(\{v\}) + f_2(\{v\})\}$ and $c(\tilde{X}) \leq B + \max_{v \in N} c(v)$.*

To extend the approach in [2], we use techniques in Murota [26] developed for M$^\natural$-concave intersection problem *without* budget constraints. An important tool for the algorithm and its analysis is a *weighted* auxiliary graph defined by local information around the current solution, while an *unweighted* auxiliary graph is used in [2]. This makes it possible, in particular, to analyze how much amount the value of the objective function changes after updating a solution.

Both of our PTASes for (kBM$^\natural$M) and (1BM$^\natural$I) are based on novel approaches in Grandoni and Zenklusen [16] and in Berger et al. [2], respectively. The adaption of these approaches in the present settings, however, are not trivial as they involve nonlinear discrete concave objective functions. The main technical contribution of this paper is to show that those previous techniques for budgeted *linear* maximization problems can be extended to budgeted *nonlinear* maximization problems by using some results in the theory of discrete convex analysis.

2 Gross Substitutes Utility and M$^\natural$-concave Functions

We give some examples of gross substitutes utility and M$^\natural$-concave functions and explain some known results. Recall the notation $a(X) = \sum_{v \in X} a(v)$ for $a \in \mathbb{R}^N$ and $X \subseteq N$.

A simple example of M$^\natural$-concave function is a linear function $f(X) = a(X)$ $(X \in \mathcal{F})$ defined on a family $\mathcal{F} \subseteq 2^N$ of matroid independent sets, where $a \in \mathbb{R}^N$. In particular, if $\mathcal{F} = 2^N$ then f is a GS utility function. Below we give some nontrivial examples. See [28, 29] for more examples of M$^\natural$-concave functions.

Example 1. (Weighted rank functions) Let $\mathcal{I} \subseteq 2^N$ be the family of independent sets of a matroid, and $w \in \mathbb{R}_+^N$. Define a function $f : 2^N \to \mathbb{R}_+$ by $f(X) = \max\{w(Y) \mid Y \subseteq X,\ Y \in \mathcal{I}\}$ $(X \in 2^N)$, which is called the *weighted rank function* [6]. If $w(v) = 1$ $(v \in N)$, then f is an ordinary rank function of the matroid $(N, \mathcal{I})$. Every weighted rank function is a GS utility function [5]. □

Example 2. (Laminar concave functions) Let $\mathcal{T} \subseteq 2^N$ be a laminar family, i.e., $X \cap Y = \emptyset$ or $X \subseteq Y$ or $X \supseteq Y$ holds for every $X, Y \in \mathcal{T}$. For $Y \in \mathcal{T}$, let $\varphi_Y : \mathbb{Z}_+ \to \mathbb{R}$ be a univariate concave function. Define a function $f : 2^N \to \mathbb{R}$ by $f(X) = \sum_{Y \in \mathcal{T}} f_Y(|X \cap Y|)$ $(X \in 2^N)$, which is called a *laminar concave function* [28, §6.3] (also called an *S-valuation* in [3]). Every laminar concave function is a GS utility function. □

Example 3. (Maximum-weight bipartite matching) Consider a bipartite graph G with two vertex sets N, J and an edge set E $(\subseteq N \times J)$, where N and J correspond to workers and jobs, respectively. Every $(u, v) \in E$ means that worker $u \in N$ has ability to process job $v \in J$, and profit $p(u, v) \in \mathbb{R}_+$ can be obtained by assigning worker u to job v. Consider a matching between workers and jobs which maximizes the total profit, and define $\mathcal{F} \subseteq 2^N$ by $\mathcal{F} = \{X \subseteq N \mid \exists M : \text{matching in } G \text{ s.t. } \partial_N M = X\}$, where $\partial_N M$ denotes the set of vertices in N covered by edges in M. It is well known that $\mathcal{F}$ is a family of independent sets in a transversal matroid. Define $f : \mathcal{F} \to \mathbb{R}$ by

$$f(X) = \max\{\textstyle\sum_{(u,v) \in M} p(u, v) \mid \exists M : \text{matching in } G \text{ s.t. } \partial_N M = X\} \ (X \in \mathcal{F}).$$

Then, f is an M$^\natural$-concave function [29, §11.4.2]. In particular, if G is a complete bipartite graph, then $\mathcal{F} = 2^N$ holds, and therefore f is a GS utility function. □

GS utility is a sufficient condition for the existence of a Walrasian equilibrium [19]; it is also a necessary condition in some sense [18]. GS utility is also related to desirable properties in the auction design (see [4, 7]); for example, an ascending item-price auction gives an approximate equilibrium, while an exact equilibrium can be computed in polynomial time (see, e.g., [22, §5]).

M$^\natural$-concave functions have various desirable properties as discrete concavity. Global optimality is characterized by local optimality, which implies the validity of a greedy algorithm for M$^\natural$-concave function maximization. Maximization of the sum of two M$^\natural$-concave functions is a nonlinear generalization of the max-weight matroid intersection problem, and can be solved in polynomial time as well. A budget constraint with uniform cost is equivalent to a cardinality constraint. Hence, (kBM$^\natural$M) and (1BM$^\natural$I) with uniform cost can be solved in polynomial time as well. The maximization of a single M$^\natural$-concave function under a general matroid constraint can be also solved exactly in polynomial time, while the corresponding problem for the sum of two M$^\natural$-concave functions is NP-hard (see [28]).

3 PTAS for *k*-budgeted M$^\natural$-concave Maximization

We prove Theorem 1.2, a key property to show the existence of a PTAS for (kBM$^\natural$M). Due to the page limitation, we mainly consider the case where f is an integer-valued function, and a more complicated proof for the general case is given in §A.5, §A.6, and §A.7 in Appendix; the proof for the integer-valued case is much simpler, but gives an idea of our algorithm for the general case.

Continuous Relaxation. Our continuous relaxation of (kBM$^\natural$M) is given by (2), where $\overline{\mathcal{F}}$ and $\overline{f}$ are defined as follows. For $X \subseteq N$ the characteristic vector of X is denoted by $\chi_X \in \{0,1\}^N$. We denote by $\overline{\mathcal{F}} \subseteq [0,1]^N$ the convex hull of vectors $\{\chi_X \mid X \in \mathcal{F}\}$, which is called a *matroid polytope*. The *concave closure* $\overline{f} : \overline{\mathcal{F}} \to \mathbb{R}$ of function f is given by

$$\overline{f}(x) = \max\left\{ \textstyle\sum_{X\in\mathcal{F}} \lambda_X f(X) \mid \sum_{X\in\mathcal{F}} \lambda_X \chi_X = x, \right. \\ \left. \textstyle\sum_{X\in\mathcal{F}} \lambda_X = 1,\ \lambda_X \geq 0\ (X \in \mathcal{F}) \right\} \quad (x \in \overline{\mathcal{F}}).$$

Note that for a general (not necessarily M$^\natural$-concave) f, the concave closure $\overline{f}$ is a polyhedral concave function satisfying $\overline{f}(\chi_X) = f(X)$ for all $X \in \mathcal{F}$. Let $S \subseteq [0,1]^N$ denote the set of feasible solutions to (CR), which is a polyhedron.

The set $P = \{(x,\alpha) \in [0,1]^N \times \mathbb{R} \mid x \in S, \alpha \leq \overline{f}(x)\}$ is a polyhedron. We say that x is a *vertex feasible solution* of (CR) if $(x, \overline{f}(x))$ is a vertex of the polyhedron P. There always exists an optimal solution of (CR) which is a vertex feasible solution, and we call such a solution a *vertex optimal solution*. Note that a vertex optimal solution does not correspond to a vertex of S in general.

Solving Continuous Relaxation. We show that if f is an integer-valued function, then (CR) can be solved exactly in polynomial time by using the ellipsoid method. Similar approach is used in Shioura [37] for the problem with a monotone M$^\natural$-concave function. We here extend the approach to the case of non-monotone M$^\natural$-concave function.

The ellipsoid method finds a vertex optimal solution of (CR) in time polynomial in n and in $\log \max_{X\in\mathcal{F}} |f(X)|$ if the following oracles are available [17]:

(O-1) polynomial-time strong separation oracle for the set S,
(O-2) polynomial-time oracle for computing a subgradient of $\overline{f}$.

The oracle (O-1) can be realized as follows. Let $x \in [0,1]^N$ be a vector. We firstly check whether the inequalities $c_i^\top x \leq B_i$ are satisfied or not. If not, then the corresponding inequality can be used as a separating hyperplane of S. We next check whether $x \in \overline{\mathcal{F}}$ or not. Recall that for a given subset $X \subseteq N$, we have an oracle to check $X \in \mathcal{F}$ in constant time. This enables us to compute the rank function $\rho : 2^N \to \mathbb{Z}_+$ of the matroid $(N, \mathcal{F})$ in polynomial time (see, e.g., [14, 28]). We have $\overline{\mathcal{F}} = \{y \in [0,1]^N \mid y(X) \leq \rho(X)\ (\forall X \in 2^N)\}$. Hence, the membership in $\overline{\mathcal{F}}$ can be checked by solving the problem $\min_{X\in 2^N}\{\rho(X) - x(X)\}$, which is a submodular function minimization and can be done in polynomial time [8, 14, 17]. Let $X_* \in \arg\min_{X\in 2^N}\{\rho(X) - x(X)\}$. If $\rho(X_*) \geq x(X_*)$, then $x \in \overline{\mathcal{F}}$ holds; otherwise, $\rho(X_*) \geq x(X_*)$ gives a separating hyperplane of S.

We then consider the oracle (O-2). A vector $p \in \mathbb{R}^N$ is called a *subgradient* of $\overline{f}$ at $x \in \overline{\mathcal{F}}$ if it satisfies $\overline{f}(y) - \overline{f}(x) \leq p^\top(y - x)$ for all $y \in \overline{\mathcal{F}}$. Lemma 1.1 (ii) states that if f is an integer-valued function, then a subgradient of $\overline{f}$ can be computed in polynomial time, i.e., the oracle (O-2) is available.

We give a proof of Lemma 1.1 (ii) by using conjugacy results of M$^\natural$-concave functions. We define a function $\overline{g} : \mathbb{R}^N \to \mathbb{R}$ by $\overline{g}(p) = \inf\{p^\top y - \overline{f}(y) \mid y \in \overline{\mathcal{F}}\}$ $(p \in \mathbb{R}^N)$. Note that $\inf\{p^\top y - \overline{f}(y) \mid y \in \overline{\mathcal{F}}\} = \inf\{p(Y) - f(Y) \mid Y \in \mathcal{F}\}$

holds, and therefore the evaluation of the function value of $\overline{g}$ can be done in polynomial time by using an M$^\natural$-concave function maximization algorithm [36]. It is well known in the theory of convex analysis (see, e.g., [35]) that $p \in \mathbb{R}^N$ is a subgradient of $\overline{f}$ at $x \in \overline{\mathcal{F}}$ if and only if $p \in \arg\max\{\overline{g}(q) - q^\top x \mid q \in \mathbb{R}^N\}$. The next lemma shows that the maximum in $\max\{\overline{g}(q) - q^\top x \mid q \in \mathbb{R}^N\}$ can be achieved by an integral vector in a finite set.

Lemma 3.1. *For every $x \in \overline{\mathcal{F}}$, there exists a subgradient p of $\overline{f}$ at x such that $p \in \mathbb{Z}^N$ and $|p(v)| \leq 2n \max_{X \in \mathcal{F}} |f(X)|$ for all $v \in N$.*

Proof of this lemma is given in §A.2 in Appendix. The discussion above and Lemma 3.1 imply that it suffices to compute an optimal solution of the problem $\max\{\overline{g}(q) - q^\top x \mid q \in \mathbb{Z}^N,\ |p(v)| \leq 2n \max_{X \in \mathcal{F}} |f(X)|\ (v \in N)\}$. The function $\overline{g}$ has a nice combinatorial structure called *L-concavity* [27, 28], and this problem can be solved exactly in time polynomial in n and in $\log \max_{X \in \mathcal{F}} |f(X)|$. Hence, we obtain the following property:

Lemma 3.2. *If f is an integer-valued function, then a vertex optimal solution of* (CR) *can be computed in polynomial time.*

Rounding of Continuous Solution. It is shown that there exists an optimal solution of (CR) which is nearly integral.

Lemma 3.3. *Let $\hat{x} \in [0,1]^N$ be a vertex optimal solution of* (CR)*. Then, $\hat{x}$ has at most $2k$ non-integral components.*

This generalizes a corresponding result in [16] for the budgeted matroid independent set problem. Below we give a proof of Lemma 3.3.

In the proof we use the concept of g-polymatorids. A *g-polymatroid* [13] is a polyhedron $Q = \{x \in \mathbb{R}^N \mid \mu(X) \leq x(X) \leq \rho(X)\ (X \in 2^N)\}$ given by a pair of submodular/supermodular functions $\rho : 2^N \to \mathbb{R} \cup \{+\infty\}$, $\mu : 2^N \to \mathbb{R} \cup \{-\infty\}$ satisfying the inequality $\rho(X) - \mu(Y) \geq \rho(X \setminus Y) - \mu(Y \setminus X)\ (X, Y \in 2^N)$. If ρ and μ are integer-valued, then Q is an integral polyhedron; in such a case, we say that Q is an *integral g-polymatroid*.

Let $\hat{x} \in [0,1]^N$ be a vertex optimal solution of (CR). Then, $\hat{x}$ is a vertex of a polyhedron given as the intersection of a set $Q = \arg\max\{\overline{f}(x) - p^\top x \mid x \in \overline{\mathcal{F}}\}$ for some $p \in \mathbb{R}^N$ and the set $K = \{x \in [0,1]^N \mid c_i^\top x \leq B_i\ (i = 1, \ldots, k)\}$. Since f is an M$^\natural$-concave function, Q is an integral g-polymatroid [31, §6]. Hence, $\hat{x}$ is contained in a d-dimensional face F of Q with $d \leq k$. The proof of the next lemma, which is a generalization of [16, Th. 3], is given in §A.3 in Appendix.

Lemma 3.4. *Let $Q \subseteq \mathbb{R}^N$ be an integral g-polymatroid and let $F \subseteq Q$ be a face of dimension d. Then, every $x \in F$ has at most $2d$ non-integral components.*

By Lemma 3.4, the number of non-integral components in $\hat{x}$ is at most $2d \leq 2k$. This concludes the proof of Lemma 3.3.

Lemma 3.3 implies the following property, stating that a solution obtained by rounding down non-integral components of a vertex optimal solution satisfies the condition in Theorem 1.2. The proof is given in §A.4 in Appendix.

Lemma 3.5. *The set* $\tilde{X} = \{v \in N \mid \hat{x}(v) = 1\}$ *is a feasible solution to* (kBM$^\natural$M) *satisfying* $f(\tilde{X}) \geq \text{OPT} - 2k \max_{v \in N} f(\{v\})$.

This, together with Lemma 3.2, implies Theorem 1.2 for integer-valued functions.

Algorithm for General Case. If the function f is not integer-valued, then it is difficult to compute a vertex optimal solution of (CR). Instead, we compute the set $\tilde{X}$ in Lemma 3.5 directly, without computing a vertex optimal solution, by using Lemma 3.3 and Lemma 3.6 below. See §A.6 and §A.7 in Appendix for the proof of Lemma 3.6 and for details of the algorithm, respectively.

Lemma 3.6. *For every fixed* $\varepsilon > 0$*, we can compute a feasible solution* $x \in [0,1]^N$ *of* (CR) *with* $\overline{f}(x) \geq (1-\varepsilon)\overline{\text{OPT}}$ *in polynomial time, where* $\overline{\text{OPT}}$ *is the optimal value of* (CR).

4 PTAS for 1-budgeted M$^\natural$-concave Intersection

We give a proof of Theorem 1.5 for (1BM$^\natural$I). With a parameter $\lambda \in \mathbb{R}_+$, the Lagrangian relaxation of (1BM$^\natural$I) is given by

$$\textbf{(LR}(\lambda)\textbf{)} \text{ Maximize } f_1(X) + f_2(X) + \lambda\{B - c(X)\} \text{ subject to } X \in \mathcal{F}_1 \cap \mathcal{F}_2.$$

The problem (LR(λ)) is an instance of the M$^\natural$-concave intersection problem *without* budget constraint, which is essentially equivalent to the valuated matroid intersection problem discussed in [25] (see §A.8 in Appendix). Therefore, the theorems and algorithms in [25] can be used to (LR(λ)) with slight modification. In particular, (LR(λ)) can be solved in polynomial time.

Below we explain how to compute a set $\tilde{X} \in \mathcal{F}_1 \cap \mathcal{F}_2$ satisfying the condition in Theorem 1.5. We firstly compute the value $\lambda = \lambda_*$ minimizing the optimal value of (LR(λ)), together with two optimal solutions X_*, Y_* of (LR(λ_*)) satisfying $c(X_*) \leq B \leq c(Y_*)$. This can be done by Megiddo's parametric search technique (see [24]; see also [2, 34]). Note that the inequality $c(X_*) \leq B \leq c(Y_*)$ implies $f_1(X_*) + f_2(X_*) \leq \text{OPT} \leq f_1(Y_*) + f_2(Y_*)$, where OPT is the optimal value of the original problem (1BM$^\natural$I). Hence, if $X_* = Y_*$ then we have $c(X_*) = B$ and $f(X_*) = \text{OPT}$, implying that $\tilde{X} = X_*$ satisfies the condition in Theorem 1.5. Otherwise (i.e., $X_* \neq Y_*$), "patching" operations are applied to X_* and Y_* to construct a better approximate solution.

The patching operations are done by using cycles in a weighted auxiliary graph. In the following, we assume that $|X_*| = |Y_*|$ holds, since in this case the description of the algorithm can be simplified (and does not lose the generality so much). We define an *auxiliary graph* $G_{X_*}^{Y_*} = (V, A)$ with arc weight $\omega : A \to \mathbb{R}$ associated with X_* and Y_* by $V = (X_* \setminus Y_*) \cup (Y_* \setminus X_*)$, $A = E_1 \cup E_2$, and

$$\begin{aligned}
E_1 &= \{(u,v) \mid u \in X_* \setminus Y_*,\ v \in Y_* \setminus X_*,\ X_* - u + v \in \mathcal{F}_1\},\\
&\quad \omega(u,v) = f_1(X_* - u + v) - f_1(X_*) + \lambda_*\{c(u) - c(v)\} \quad ((u,v) \in E_1),\\
E_2 &= \{(v,u) \mid v \in Y_* \setminus X_*,\ u \in X_* \setminus Y_*,\ X_* + v - u \in \mathcal{F}_2\},\\
&\quad \omega(v,u) = f_2(X_* + v - u) - f_2(X_*) \quad ((v,u) \in E_2).
\end{aligned}$$

A cycle in $G_{X_*}^{Y_*}$ is a directed closed path which visits each vertex at most once. In every cycle in $G_{X_*}^{Y_*}$, arcs in E_1 and arcs in E_2 appear alternately, and every cycle contains an even number of arcs.

For a cycle C in the graph $G_{X_*}^{Y_*}$, we define a set $X_* \oplus C$ $(\subseteq N)$ by

$$X_* \oplus C = X_* \setminus \{u \in X_* \setminus Y_* \mid (u,v) \in C \cap E_1\} \cup \{v \in Y_* \setminus X_* \mid (u,v) \in C \cap E_1\}.$$

Lemma 4.1. (i) *A maximum-weight cycle in $G_{X_*}^{Y_*}$ is a zero-weight cycle.*
(ii) *Let C be a zero-weight cycle in $G_{X_*}^{Y_*}$ with the minimum number of arcs. Then, $X_* \oplus C$ is an optimal solution of* (LR(λ_*)) *with $X_* \oplus C \neq X_*$.*

Lemma 4.1 (i) implies that a zero-weight cycle C in $G_{X_*}^{Y_*}$ with the minimum number of arcs can be computed by using a shortest-path algorithm. The set $X' = X_* \oplus C$ is an optimal solution of (LR(λ_*)) by Lemma 4.1 (ii). If $X' = Y_*$, then an additional patching operation explained below is applied. If $c(X') = B$, then we stop since X' satisfies the condition in Theorem 1.5. If $c(X') < B$ then we replace X_* with X'; otherwise (i.e., $c(X') > B$), we replace Y_* with X'; in both cases, we repeat the same patching operations.

We explain the additional patching operation in the case where $X_* \oplus C = Y_*$. In this case, C contains all vertices in the graph $G_{X_*}^{Y_*}$. Let $a_1, a_2, \ldots, a_{2h} \in A$ be a sequence of arcs in the cycle C, where $2h$ is the number of arcs in C. We may assume that $a_j \in E_1$ if j is odd and $a_j \in E_2$ if j is even. For $j = 1, 2, \ldots, h$, let $\alpha_j = \omega(a_{2j-1}) + \omega(a_{2j})$. Since C is a zero-weight cycle, we have $\sum_{j=1}^{h} \alpha_j = 0$.

Lemma 4.2 (Gasoline Lemma (cf. [23])). *Let $\alpha_1, \alpha_2, \ldots, \alpha_h$ be real numbers satisfying $\sum_{j=1}^{h} \alpha_j = 0$. Then, there exists some $t \in \{1, \ldots, h\}$ such that $\sum_{j=t}^{t+i} \alpha_{j (\mathrm{mod}\, h)} \geq 0$ $(i = 0, 1, \ldots, h{-}1)$, where $\alpha_0 = \alpha_h$.*

By Lemma 4.2, we may assume that $\sum_{j=1}^{i} \alpha_j \geq 0$ for all $i = 1, 2, \ldots, h$. For $j = 1, 2, \ldots, h$, we denote $a_{2j-1} = (u_j, v_j)$, and let $\eta_j = c(v_j) - c(u_j)$. Then, $c(Y_*) = c(X_*) + \sum_{j=1}^{h} \eta_j$ holds. Let $t \in \{1, 2, \ldots, h\}$ be the minimum integer such that $c(X_*) + \sum_{j=1}^{t} \eta_j > B$. Since $c(X_*) < B$, we have $t \geq 1$. In addition, the choice of t implies that $c(X_*) + \sum_{j=1}^{t-1} \eta_j \leq B$. With the arc set $C' = \{a_1, a_2, \ldots, a_{2t-1}, a_{2t}\}$, we define $\tilde{X} \subseteq N$ by

$$\tilde{X} = X_* \setminus \{u \in X_* \mid (u,v) \in C' \cap E_1 \text{ or } u = u_{t+1}\} \cup \{v \in N \setminus X_* \mid (u,v) \in C' \cap E_1\}.$$

We show that the set $\tilde{X}$ satisfies the desired condition in Theorem 1.5.

We have

$$\begin{aligned}
\textsc{opt} &\leq f_1(X_*) + f_2(X_*) + \lambda_* \{B - c(X_*)\} + \textstyle\sum_{j=1}^{t} \alpha_j \\
&\leq [f_1(X_*) + \textstyle\sum_{j=1}^{t} \{f_1(X_* - u_j + v_j) - f_1(X_*)\}] \\
&\qquad + [f_2(X_*) + \textstyle\sum_{j=1}^{t} \{f_1(X_* - u_{j+1} + v_j) - f_1(X_*)\}].
\end{aligned}$$

We define $\tilde{X}_1 = \tilde{X} \cup \{u_{t+1}\}$ and $\tilde{X}_2 = \tilde{X} \cup \{u_1\}$. Note that $\tilde{X}_1 \cap \tilde{X}_2 = \tilde{X}$. By using the fact that C' is a subpath of a zero-weight cycle with the smallest number of arcs, we can show the following:

Lemma 4.3. *We have* $\tilde{X}_1 \in \mathcal{F}_1$, $\tilde{X}_2 \in \mathcal{F}_2$, $f_1(\tilde{X}_1) = f_1(X_*) + \sum_{j=1}^{t}\{f_1(X_* - u_j + v_j) - f_1(X_*)\}$, *and* $f_2(\tilde{X}_2) = f_2(X_*) + \sum_{j=1}^{t}\{f_1(X_* - u_{j+1} + v_j) - f_1(X_*)\}$.

Hence, we obtain $f_1(\tilde{X}_1) + f_2(\tilde{X}_2) \geq \text{OPT}$. M$^\natural$-concavity of f_1 and f_2 implies

$$\begin{aligned} f_1(\tilde{X}) + f_2(\tilde{X}) &\geq f_1(\tilde{X}_1) - \{f_1(\tilde{X}_1) - f_1(\tilde{X})\} + f_2(\tilde{X}_2) - \{f_2(\tilde{X}_2) - f_2(\tilde{X})\} \\ &\geq f_1(\tilde{X}_1) - \{f_1(\{u_{t+1}\}) - f_1(\emptyset)\} + f_2(\tilde{X}_2) - \{f_2(\{u_1\}) - f_2(\emptyset)\} \\ &\geq \text{OPT} - 2 \cdot \max_{v \in N}\{f_1(\{v\}) + f_2(\{v\})\}, \end{aligned}$$

from which the former inequality in Theorem 1.5 follows. The latter inequality in Theorem 1.5 can be shown as follows:

$$\begin{aligned} c(\tilde{X}) &= c(X_*) + \textstyle\sum_{j=1}^{t} \eta_j - c(u_{t+1}) \\ &\leq \{c(X_*) + \textstyle\sum_{j=1}^{t-1} \eta_j\} + \eta_t \leq B + \eta_t \leq B + \max_{v \in N} c(v). \end{aligned}$$

References

1. L. M. Ausubel and P. Milgrom. Ascending auctions with package bidding. Front. Theor. Econ., 1 (2002) Article 1.
2. A. Berger, V. Bonifaci, F. Grandoni, and G. Schäfer. Budgeted matching and budgeted matroid intersection via the gasoline puzzle. Math. Programming (2009), to appear.
3. M. Bing, D. Lehmann, and P. Milgrom. Presentation and structure of substitutes valuations. Proc. EC 2004, 238–239.
4. L. Blumrosen and N. Nisan. Combinatorial auction. in: N. Nisan, T. Roughgarden, É. Tardos, and V. V. Vazirani (eds.) Algorithmic Game Theory, pp. 267–299. Cambridge Univ. Press, 2007.
5. G. Calinescu, C. Chekuri, M. Pál, and J. Vondrák. Maximizing a submodular set function subject to a matroid constraint. Proc. IPCO 2007, 182–196.
6. G. Calinescu, C. Chekuri, M. Pál, and J. Vondrák. Maximizing a submodular set function subject to a matroid constraint. SIAM J. Comput., to appear.
7. P. Cramton, Y. Shoham, and R. Steinberg. Combinatorial Auctions. MIT Press, 2006.
8. W. H. Cunningham. On submodular function minimization. Combinatorica 5 (1985) 185–192.
9. A.W.M. Dress and W. Wenzel. Valuated matroids. Adv. Math. 93 (1992) 214–250.
10. U. Feige. A threshold of $\ln n$ for approximating set cover. J. ACM 45 (1998) 634-652.
11. U. Feige. On maximizing welfare when utility functions are subadditive. SIAM J. Comput. 39 (2009) 122–142.
12. U. Feige, V. S. Mirrokni, and J. Vondrák. Maximizing non-monotone submodular functions. Proc. FOCS 2007, 461–471.
13. A. Frank and É. Tardos. Generalized polymatroids and submodular flows. Math. Programming 42 (1988) 489–563.
14. S. Fujishige. Submodular Functions and Optimization. 2nd Edition. Elsevier, 2005.
15. S. Fujishige and Z. Yang. A note on Kelso and Crawford's gross substitutes condition. Math. Oper. Res. 28 (2003) 463–469.

16. F. Grandoni and R. Zenklusen. Approximation schemes for multi-budgeted independence systems. Proc. ESA 2010, 536–548.
17. M. Grötschel, L. Lovász, and A. Schrijver. Geometric algorithms and combinatorial optimization. 2nd Edition. Springer, 1993.
18. F. Gul and E. Stacchetti. Walrasian equilibrium with gross substitutes. J. Econ. Theory 87 (1999) 95–124.
19. A. S. Kelso and V. P. Crawford. Job matching, coalition formation and gross substitutes. Econometrica 50 (1982) 1483–1504.
20. A. Kulik, H. Shachnai, and T.Tamir. Maximizing submodular set functions subject to multiple linear constraints. Proc. SODA 2009, 545–554.
21. J. Lee, V. S. Mirrokni, V. Nagarajan, and M. Sviridenko. Maximizing nonmonotone submodular functions under matroid or knapsack constraints. SIAM J. Discrete Math. 23 (2010) 2053–2078.
22. B. Lehmann, D. Lehmann, and N. Nisan. Combinatorial auctions with decreasing marginal utilities. Games Econom. Behav. 55 (2006) 270–296.
23. S. Lin and B.W. Kernighan. An effective heuristic algorithm for the traveling salesman problem. Oper. Res. 21 (1973) 498–516.
24. N. Megiddo. Combinatorial optimization with rational objective functions. Math. Oper. Res. 4 (1979) 414–424.
25. K. Murota. Convexity and Steinitz's exchange property. Adv. Math. 124 (1996) 272–311.
26. K. Murota. Valuated matroid intersection, I: optimality criteria, II: algorithms. SIAM J. Discrete Math. 9 (1996) 545-561, 562-576.
27. K. Murota. Discrete convex analysis. Math. Programming 83 (1998) 313–371.
28. K. Murota. Discrete Convex Analysis. SIAM, Philadelphia, 2003.
29. K. Murota. Recent developments in discrete convex analysis. in: W. J. Cook, J. Lovász, and J. Vygen (eds.) Research Trends in Combinatorial Optimization, pp. 219–260. Springer, 2009.
30. K. Murota and A. Shioura. M-convex function on generalized polymatroid. Math. Oper. Res. 24 (1999) 95–105.
31. K. Murota and A. Shioura. Extension of M-convexity and L-convexity to polyhedral convex functions. Adv. in Appl. Math. 25 (2000) 352–427.
32. K. Murota and A. Tamura. Application of M-convex submodular flow problem to mathematical economics. Japan J. Indust. Appl. Math. 20 (2003) 257–277.
33. K. Murota and A. Tamura. New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities. Discrete Appl. Math. 131 (2003) 495–512.
34. R. Ravi and M. X. Goemans. The constrained minimum spanning tree problem. Proc. SWAT 1996, 66–75.
35. R.T. Rockafellar. Convex Analysis. Princeton Univ. Press, 1970.
36. A. Shioura. Minimization of an M-convex function. Discrete Appl. Math. 84 (1998) 215–220.
37. A. Shioura. On the pipage rounding algorithm for submodular function maximization: a view from discrete convex analysis. Discrete Math. Algorithms Appl. 1 (2009) 1–23.
38. M. Sviridenko. A note on maximizing a submodular set function subject to a knapsack constraint. Oper. Res. Lett. 32 (2004) 41–43.
39. L.A. Wolsey. Maximising real-valued submodular functions: primal and dual heuristics for location problems. Math. Oper. Res. 7 (1982) 410–425.

A Appendix

A.1 Partial Enumeration Technique for PTAS

Theorems 1.2 and 1.5 state that there exist polynomial-time algorithms which compute high-quality feasible solutions for (kBMM) and for (1BMI), respectively. We show that by using a standard technique called "partial enumeration" (see, e.g., [2, 16, 34]), these algorithms can be transformed into PTASes for (kBMM) and for (1BMI), respectively.

We here consider a more general setting. Let $\mathcal{F} \subseteq 2^N$ be an independence system, i.e., if $X \in \mathcal{F}$ and $Y \subseteq X$ then $Y \in \mathcal{F}$. Let $f : \mathcal{F} \to \mathbb{R}$ be a submodular function defined on $\mathcal{F}$, i.e.,

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \ (\forall X, Y \in \mathcal{F} \text{ s.t. } X \cap Y, X \cup Y \in \mathcal{F}). \quad (3)$$

For $X \in \mathcal{F}$ and $Y \subseteq N$ with $X \subseteq Y$, we define a set family $\mathcal{F}_X^Y$ ($\subseteq 2^{Y \setminus X}$) and a function $f_X^Y : \mathcal{F}_X^Y \to \mathbb{R}$ by

$$\mathcal{F}_X^Y = \{U \mid U \subseteq Y \setminus X,\ U \cup X \in \mathcal{F}\}, \quad (4)$$

$$f_X^Y(U) = f(X \cup U) - f(X) \quad (U \in \mathcal{F}_X^Y). \quad (5)$$

Note that $\mathcal{F}_X^Y$ is an independence system and f_X^Y is a submodular function on $\mathcal{F}_X^Y$ with $f_X^Y(\emptyset) = 0$. We say that $\mathcal{F}_X^Y$ (resp., f_X^Y) is a *minor* of $\mathcal{F}$ (resp., f).

Let $\mathcal{S}$ be a family of submodular functions $f : \mathcal{F} \to \mathbb{R}$ defined on independence systems $\mathcal{F}$ such that $f(\emptyset) = 0$, and assume that $\mathcal{S}$ is *minor-closed*, i.e., every minor of $f \in \mathcal{S}$ is also in $\mathcal{S}$. We consider the following budgeted optimization problem:

$$(k\text{BSM}) \text{ Maximize } \ f(X) \quad \text{ subject to } \ X \in \mathcal{F},\ c_i(X) \leq B_i \ (1 \leq i \leq k), \quad (6)$$

where $f : \mathcal{F} \to \mathbb{R}$ is a function in $\mathcal{S}$, k is a constant positive integer, and $c_i \in \mathbb{R}_+^N$ and $B_i \in \mathbb{R}_+$ for $i = 1, 2, \ldots, k$. We denote by OPT the optimal value of (kBSM). Note that the problems (kBMM) and (1BMI) are special cases of (kBSM). We may assume that $f(\{v\}) > 0$ for all $v \in N$; if $f(\{v\}) \leq 0$ then the submodularity of f implies that there exists an optimal solution which does not contain v, and therefore the element v can be ignored.

We prove the following theorem by applying the partial enumeration technique to (kBSM).

Theorem A.1. *Let $\alpha \in [0, 1]$ and $\eta \in \mathbb{Z}_+$. Suppose that the problem* (kBSM) *has a polynomial-time algorithm which computes a set $\tilde{X} \in \mathcal{F}$ satisfying*

$$f(\tilde{X}) \geq \alpha \cdot \text{OPT} - \eta \cdot \max_{v \in N} f(\{v\}),$$

$$c_i(\tilde{X}) \leq B_i + \eta \cdot \max_{v \in N} c_i(v) \ (i = 1, 2, \ldots, k).$$

Then, (kBSM) *has a polynomial-time $(\alpha - \varepsilon)$-approximation algorithm for every fixed constant $\varepsilon \in (0, \alpha]$.*

Then, Theorem 1.3 (resp., Theorem 1.4) is an immediate consequence of Theorem A.1 and Theorem 1.2 (resp., Theorem 1.5), where $\alpha = 1$ and $\eta = 2k$ (resp., $\eta = 2$).

We now give a proof of Theorem A.1. Let $\varepsilon' = \varepsilon/(\alpha+1)$, and $X_* \in \mathcal{F}$ be an optimal solution of (kBSM) which is fixed in the following discussion. We may assume that $|X_*| > (k+1)\eta/\varepsilon'$ since otherwise the cardinality of X_* is bounded by a constant number and therefore such X_* can be found by a brute-force algorithm in polynomial time.

Our algorithm consists of the following three major steps:

Step 1: Guess a subset X_b of X_* with $|X_b| = (k+1)\eta/\varepsilon'$ consisting of "large" elements.

Step 2: Compute a set X_s satisfying the following condition

$$\left.\begin{array}{l} X_b \cup X_s \in \mathcal{F}, \\ f(X_b \cup X_s) \geq (\alpha - \varepsilon')\textsc{opt}, \\ c_i(X_b \cup X_s) \leq (1+\varepsilon')B_i \ (i = 1, 2, \ldots, k). \end{array}\right\} \tag{7}$$

Step 3: Compute a subset U of $X_b \cup X_s$ such that $(X_b \cup X_s) \setminus U$ is an $(1-\varepsilon')(\alpha-\varepsilon')$-approximate feasible solution to (kBSM).

Note that the output in Step 3 is an $(\alpha-\varepsilon)$-approximate solution since $(1-\varepsilon')(\alpha-\varepsilon') \geq \alpha - \varepsilon$.

We firstly guess a subset Z_0 of X_* with $|Z_0| = \eta/\varepsilon'$ which maximizes the value $f(Z_0)$. Here, we assume, for simplicity and without loss of generality, that η/ε' is an integer. This is done by enumerating all subsets of N with cardinality η/ε'. Since η/ε' is a constant, this can be done in polynomial time.

Let

$$N_0 = \{v \in N \setminus Z_0 \mid f(Z_0 \cup \{v\}) - f(Z_0) \leq (\varepsilon'/\eta) f(Z_0)\}.$$

We claim that $X_* \setminus Z_0 \subseteq N_0$ holds. Assume, to the contrary, that there exists some $v \in X_* \setminus Z_0$ which is not in N_0. It follows from the submodularity (3) of f that

$$\begin{aligned} \min_{u \in X_0} \{f(Z_0) - f(Z_0 \setminus \{u\})\} &\leq \frac{1}{|Z_0|} \sum_{u \in Z_0} \{f(Z_0) - f(Z_0 \setminus \{u\})\} \\ &\leq \frac{1}{|Z_0|} \{f(Z_0) - f(\emptyset)\} \ = \ \frac{\varepsilon'}{\eta} f(Z_0). \end{aligned} \tag{8}$$

For every $u \in X_0$, we have

$$f((Z_0 \setminus \{u\}) \cup \{v\}) - f(Z_0 \setminus \{u\}) \geq f(Z_0 \cup \{v\}) - f(Z_0) > \frac{\varepsilon'}{\eta} f(Z_0), \tag{9}$$

where the first inequality is by submodularity (3) and the second follows from $v \notin N_0$. Combining (8) and (9), we obtain

$$f(Z_0) \leq f(Z_0 \setminus \{u_*\}) + \frac{\varepsilon'}{\eta} f(Z_0) < f((Z_0 \setminus \{u_*\}) \cup \{v\}),$$

where $u = u_* \in X_0$ minimizes the value $f(Z_0) - f(Z_0 \setminus \{u\})$. This, however, is a contradiction to the choice of Z_0. Hence, $X_* \setminus Z_0 \subseteq N_0$ holds.

Next, for $i = 1, 2, \ldots, k$, we iteratively guess a set Z_i of η/ε' largest elements in $X_* \setminus (Z_0 \cup Z_1 \cup \cdots \cup Z_{i-1})$ with respect to the cost $c_i(v)$, and let

$$N_i = \{v \in N_{i-1} \setminus Z_i \mid c_i(v) \leq \min_{u \in Z_i} c_i(u)\}.$$

Then, we have $X_* \setminus (Z_0 \cup Z_1 \cup \cdots \cup Z_{i-1} \cup Z_i) \subseteq N_i$. It should be noted that guessing the sets $Z_1, \ldots, Z_k$ can be done in polynomial time since k and η/ε' are constant numbers. We put $X_b = \cup_{i=0}^k Z_i$ and $Y = X_b \cup N_k$. Note that X_b is a feasible solution to (1).

We claim that if $X_b \subseteq X_*$ then X_b is a feasible solution to (kBSM) with $f(X_b) \geq 0$. The feasibility of X_b is easy to see, and the inequality follows from $f(X_b) \geq f(X_*) - f(X_* \setminus X_b) \geq 0$, where the first inequality is by the submodularity of f and the second by the optimality of X_*. This means that if X_b is not feasible to (kBSM) or $f(X_b) < 0$ holds, then some of the guesses for $Z_0, Z_1, \ldots, Z_k$ are not done correctly, and therefore the current X_b should be ignored. In the following, we may assume that X_b is a feasible solution to (kBSM) with $f(X_b) \geq 0$.

We then consider Step 2. We denote $\mathcal{F}' = \mathcal{F}^Y_{X_b}$ ($\subseteq 2^{N_k}$) and $f' = f^Y_{X_b}$ (see (4) and (5) for the definitions of $\mathcal{F}^Y_{X_b}$ and $f^Y_{X_b}$). Then, f' is a function defined on $\mathcal{F}'$ with $f' \in \mathcal{S}$. We consider an instance of (kBSM) given by

$$\text{Maximize} \quad f'(U) \quad \text{subject to} \quad U \in \mathcal{F}', \ c_i(U) \leq B'_i \ (1 \leq i \leq k),$$

where $B'_i = B_i - c_i(X_b)$ for each i. We denote by $\textsc{opt}'$ the optimal value of this instance. Then, $\textsc{opt}' + f(X_b) = \textsc{opt}$ holds, provided that the sets $Z_0, Z_1, \ldots, Z_k$ are guessed correctly. The assumption of Theorem A.1 implies that in polynomial time we can compute a set $X_s \in \mathcal{F}'$ satisfying

$$f'(X_s) \geq \alpha \cdot \textsc{opt}' - \eta \cdot \max_{v \in N_k} f'(\{v\}), \tag{10}$$

$$c_i(X_s) \leq B'_i + \eta \cdot \max_{v \in N_k} c_i(v) \qquad (i = 1, 2, \ldots, k). \tag{11}$$

We show that the set $X_b \cup X_s$ satisfies the three conditions in (7). Since $X_s \in \mathcal{F}'$, we have $X_b \cup X_s \in \mathcal{F}$, i.e., the first condition in (7) holds.

By the submodularity (3) of f, $Z_0 \subseteq X_b$, and $N_k \subseteq N_0$, we have

$$\begin{aligned} \max_{v \in N_k} f'(\{v\}) &= \max_{v \in N_k} \{f(X_b \cup \{v\}) - f(X_b)\} \\ &\leq \max_{v \in N_k} \{f(Z_0 \cup \{v\}) - f(Z_0)\} \\ &\leq \max_{v \in N_0} \{f(Z_0 \cup \{v\}) - f(Z_0)\} \ \leq \ \frac{\varepsilon'}{\eta} f(Z_0). \end{aligned} \tag{12}$$

By the choice of Z_i and $N_k \subseteq N_i$, we have

$$\max_{v \in N_k} c_i(v) \leq \max_{v \in N_i} c_i(v) \leq \min_{u \in Z_i} c_i(u) \leq \frac{1}{|Z_i|} c_i(Z_i) = \frac{\varepsilon'}{\eta} c_i(Z_i). \tag{13}$$

Hence, it follows that

$$\begin{aligned} f(X_b \cup X_s) &= f'(X_s) + f(X_b) \\ &\geq \alpha \cdot \text{OPT}' - \eta \cdot \max_{v \in N_k} f'(\{v\}) + f(X_b) \\ &= \alpha \cdot \text{OPT} + (1-\alpha) f(X_b) - \eta \cdot \max_{v \in N_k} f'(\{v\}) \\ &\geq \alpha \cdot \text{OPT} + (1-\alpha) f(X_b) - \eta \cdot \frac{\varepsilon'}{\eta} f(Z_0) \\ &\geq \alpha \cdot \text{OPT} - \varepsilon' f(Z_0) \\ &\geq (\alpha - \varepsilon')\text{OPT}, \end{aligned} \tag{14}$$

where the first inequality is by (10), the second by (12), the third by $f(X_b) \geq 0$, and the fourth by $f(Z_0) \leq \text{OPT}$. Hence, the second condition in (7) holds.

It follows from (11), (13), and $c_i(Z_i) \leq B_i$ that

$$\begin{aligned} c_i(X_b \cup X_s) &\leq B_i + \eta \cdot \max_{v \in N_k} c_i(v) \\ &\leq B_i + \varepsilon' c_i(Z_i) \leq (1+\varepsilon')B_i \quad (i = 1, 2, \ldots, k). \end{aligned} \tag{15}$$

That is, the third condition in (7) holds.

In Step 3, we finally construct an $(1-\varepsilon')(\alpha-\varepsilon')$-approximate feasible solution by deleting some elements in $X_b \cup X_s$. We assume, for simplicity, that $1/\varepsilon'$ is an integer. Let $\{U_1, U_2, \ldots, U_{(1/\varepsilon')-1}, U_{1/\varepsilon'}\}$ be an arbitrarily chosen partition of X_s such that $|U_j \cap Z_h| = \eta$ for each j and h; recall that $|Z_h| = \eta/\varepsilon'$ for all $h = 0, 1, \ldots, k$. We also set $t = (1/\varepsilon') + 1$ and $U_t = X_s$. Then, $\{U_1, U_2, \ldots, U_t\}$ is a partition of $X_b \cup X_s$.

It is easy to see that $(X_b \cup X_s) \setminus U_t = X_b$ is a feasible solution to (kBSM). For each $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, 1/\varepsilon'$, it holds that

$$c_i(U_j) \geq c_i(U_j \cap Z_i) \geq \eta \cdot \min_{u \in Z_i} c_i(u) \geq \eta \cdot \max_{v \in N_i} c_i(v) \geq \eta \cdot \max_{v \in N_k} c_i(v)$$

since $N_k \subseteq N_i$. By (15), we have

$$\begin{aligned} c_i((X_b \cup X_s) \setminus U_j) &= c_i(X_b \cup X_s) - c_i(U_j) \\ &\leq B_i + \eta \cdot \max_{v \in N_k} c_i(v) - \eta \cdot \max_{v \in N_k} c_i(v) = B_i. \end{aligned}$$

Hence, $(X_b \cup X_s) \setminus U_j$ is a feasible solution to (kBSM) for each $j = 1, 2, \ldots, t-1$. Recall that $(X_b \cup X_s) \setminus U_t = X_b$ is also feasible solution to (kBSM).

To conclude the proof of Theorem A.1, we show that the following inequality holds:

$$\max_{1 \leq j \leq t} f((X_b \cup X_s) \setminus U_j) \geq (1-\varepsilon')(\alpha - \varepsilon')\text{OPT}.$$

This can be shown by using the following property of f:

Lemma A.1 (cf. [11]). *Let $f : \mathcal{F} \to \mathbb{R}$ be a function defined on an independence system $\mathcal{F} \subseteq 2^N$, and suppose that f is submodular in the sense of* (3)*.*

Then, for every $U \in \mathcal{F}$ and a positive integer t, it holds that

$$f(U) \le \sum_{j=1}^{t} \lambda_j f(V_j)$$

for any t subsets $V_1, V_2, \ldots, V_t$ of U and nonnegative real numbers λ_j ($j = 1, 2, \ldots, t$) such that $\sum_{j=1}^{t} \lambda_j = 1$ and $\sum_{j=1}^{t} \lambda_j \chi_{V_j} = \chi_U$.

By Lemma A.1 and (14), it holds that

$$\begin{aligned}\max_{1 \le j \le t} f((X_b \cup X_s) \setminus U_j) &\ge \frac{1}{t} \sum_{j=1}^{t} f((X_b \cup X_s) \setminus U_j) \\ &\ge \frac{1}{t} \cdot (t-1) f(X_b \cup X_s) \\ &\ge (1-\varepsilon') f(X_b \cup X_s) \ \ge \ (1-\varepsilon')(\alpha - \varepsilon')\text{OPT}.\end{aligned}$$

This concludes the proof of Theorem A.1.

A.2 Proof of Lemma 3.1

Let $x \in \overline{\mathcal{F}}$. Recall that $\overline{f}$ is a polyhedral concave function defined on $\overline{\mathcal{F}}$, and by assumption $\overline{\mathcal{F}}$ is a full-dimensional polytope. Therefore, there exists a subgradient $p \in \mathbb{R}^N$ at x such that the set

$$D = \{y \in \overline{\mathcal{F}} \mid \overline{f}(y) - \overline{f}(x) = p^\top (y - x)\}$$

is a full-dimensional polytope. We show that such a subgradient p satisfies the inequality

$$|p(v)| \le 2n \max_{X \in \mathcal{F}} |f(X)| \qquad (v \in N). \tag{16}$$

The set D can be represented as

$$D = \arg\max\{\overline{f}(y) - p^\top y \mid y \in \overline{\mathcal{F}}\}.$$

Since f is an M$^\natural$-concave function, D is an integral g-polymatroid with $D \subseteq [0,1]^N$ (cf. [31, §6]). Let x_0 be a vertex of D, which is a 0-1 vector corresponding to some set $X_0 \subseteq N$. We consider the tangent cone of D at x_0, which is generated by the set W of the following vectors (cf. [14, Th. 3.28]):

$$\begin{aligned}&+\chi_v \ (v \in N,\ x_0 + \chi_v \in D), \quad -\chi_v \ (v \in N,\ x_0 - \chi_v \in D),\\ &+\chi_u - \chi_v \ (u, v \in N,\ x_0 + \chi_u - \chi_v \in D).\end{aligned}$$

Since D is full-dimensional, its tangent cone is also full-dimensional, which implies that W contains n linear independent vectors. Hence, the vector p is a (unique) solution of the system of the following linear equations, where $q \in \mathbb{R}^N$ is a variable vector:

$$\begin{aligned}
&+q(v) = \overline{f}(x_0 + \chi_v) - \overline{f}(x_0)\ (= f(X_0 + v) - f(X_0)) \quad (v \in N,\ x_0 + \chi_v \in D),\\
&-q(v) = \overline{f}(x_0 - \chi_v) - \overline{f}(x_0)\ (= f(X_0 - v) - f(X_0)) \quad (v \in N,\ x_0 - \chi_v \in D),\\
&+q(u) - q(v) = \overline{f}(x_0 + \chi_u - \chi_v) - \overline{f}(x_0)\ (= f(X_0 + u - v) - f(X_0))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (u, v \in N,\ x_0 + \chi_u - \chi_v \in D).
\end{aligned}$$

Recall that for every $X \in \mathcal{F}$ we have $f(X) = \overline{f}(\chi_X)$. Since the coefficient matrix of this system of linear equations is totally unimodular, this system has an integral solution, i.e., $p \in \mathbb{Z}^N$.

The linear equations above imply that

$$|p(v)| \le 2 \max_{X \in \mathcal{F}} |f(X)| \quad (v \in N,\ x_0 + \chi_v \in D \text{ or } x_0 - \chi_v \in D), \tag{17}$$

$$|p(u) - p(v)| \le 2 \max_{X \in \mathcal{F}} |f(X)| \quad (u, v \in N,\ x_0 + \chi_u - \chi_v \in D). \tag{18}$$

From these inequalities we derive the inequality (16), where we use an undirected graph $G = (N, E)$ with the vertex set N and the edge set

$$E = \{(u, v) \mid u, v \in N,\ x_0 + \chi_u - \chi_v \in D\}.$$

In addition, we define $R = \{v \in N \mid x_0 + \chi_v \in D \text{ or } x_0 - \chi_v \in D\}$, and call each element in R a root vertex. By using the fact that W contains n linear independent vectors, we can show that for every vertex v in G, there exists a path from v to some root vertex $r \in R$. Let $v_0 = v, v_1, v_2, \ldots, v_k = r$ be the sequence of the vertices in such a path between v and r, where $k \le n - 1$. By (17) and (18), it holds that

$$\begin{aligned}
|p(v)| &\le |p(v_0) - p(v_1)| + |p(v_1) - p(v_2)| + \cdots + |p(v_{k-1}) - p(v_k)| + |p(v_k)|\\
&\le 2(k+1) \max_{X \in \mathcal{F}} |f(X)| \ \le\ 2n \max_{X \in \mathcal{F}} |f(X)|.
\end{aligned}$$

A.3 Proof of Lemma 3.4

To prove Lemma 3.4, we use the concept of base polyhedron [14] which is deeply related to the concept of g-polymatroid. A *base polyhedron* is a polyhedron $S \subseteq \mathbb{R}^N$ given by $S = \{x \in \mathbb{R}^N \mid x(X) \le \rho(X)\ (X \subseteq N),\ x(N) = \rho(N)\}$ with a submodular function $\rho : 2^N \to \mathbb{R} \cup \{+\infty\}$ with $\rho(\emptyset) = 0$ and $\rho(N) < +\infty$. If ρ is integer-valued, then S is an integral polyhedron; in such a case, we say that Q is an *integral base polyhedron*. It is shown that a polyhedron $Q \subseteq \mathbb{R}^N$ is a g-polymatroid if and only if the set

$$\tilde{Q} = \{(-x(N), x) \in \mathbb{R}^{\{0\} \cup N} \mid x \in Q\}, \tag{19}$$

where 0 is a new element not in N, is a base polyhedron (see [14]). Lemma 3.4 for g-polymatroids can be restated in terms of base polyhedra as follows.

Lemma A.2. *Let $S \subseteq \mathbb{R}^N$ be an integral base polyhedron associated with an integer-valued submodular function $\rho : 2^N \to \mathbb{Z} \cup \{+\infty\}$ satisfying $\rho(\emptyset) = 0$ and $\rho(N) < +\infty$, and $F \subseteq S$ be a face of dimension d. Then, every $x \in F$ has at most $2d$ non-integral components.*

Proof. Since the dimension of F is d and every $x \in F$ satisfies $x(N) = \rho(N)$, there exist $n - d - 1$ distinct sets $Y_1, Y_2, \dots, Y_{n-d-1} \subset N$ such that

$$F = \{x \in S \mid x(Y_j) = \rho(Y_j) \ (j = 1, 2, \dots, n - d)\},$$

where $Y_{n-d} = N$. By a standard uncrossing argument (see, e.g., [14, 17]), we can assume that $\emptyset \neq Y_1 \subset Y_2 \subset \cdots \subset Y_{n-d} = N$ holds. Let $\hat{x} \in \mathbb{R}^N$ be an arbitrarily chosen vector in F. Putting $S_j = Y_j \setminus Y_{j-1}$ $(\neq \emptyset)$ $(j = 1, 2, \dots, n - d)$, it holds that $\hat{x}(S_j) = \rho(Y_j) - \rho(Y_{j-1}) \in \mathbb{Z}$, where $Y_0 = \emptyset$. This implies that if $|S_j| = 1$ then $\hat{x}(v) \in \mathbb{Z}$ for the unique element v in S_j. Since $|N| = n$, at least $n - 2d$ sets among $S_1, S_2, \dots, S_{n-d}$ are singleton sets. Hence, $\hat{x}$ has at most $2d$ non-integral components. □

Lemma 3.4 can be shown as follows. Let $Q \subseteq \mathbb{R}^N$ be an integral g-polymatroid and $F \subseteq Q$ be a d-dimensional face of Q. We define an integral base polyhedron $\tilde{Q} \subseteq \mathbb{R}^{\{0\} \cup N}$ as in (19). Then, the set $\tilde{F} = \{(-x(N), x) \in \mathbb{R}^{\{0\} \cup N} \mid x \in F\}$ is a d-dimensional face of $\tilde{Q}$. By Lemma A.2, every $\tilde{x} \in \tilde{F}$ has at most $2d$ non-integral components. Hence, every $x \in F$ has at most $2d$ non-integral components.

A.4 Proof of Lemma 3.5

We prove that the set $\tilde{X} = \{v \in N \mid \hat{x}(v) = 1\}$ is a feasible solution to (kBM$^\natural$M) satisfying $f(\tilde{X}) \geq \text{OPT} - 2k \max_{v \in N} f(\{v\})$.

Recall that $\hat{x} \in [0, 1]^N$ is a vertex optimal solution of (CR). Let $\tilde{x} \in \{0, 1\}^N$ be a vector obtained by rounding down the non-integral components of $\hat{x}$, i.e., $\tilde{x}(v) = 1$ if $\hat{x}(v) = 1$ and $\tilde{x}(v) = 0$ otherwise. Note that $\tilde{x}$ is the characteristic vector of $\tilde{X}$ and therefore satisfies $\overline{f}(\tilde{x}) = f(\tilde{X})$.

We firstly show that $\tilde{X}$ is a feasible solution to (kBM$^\natural$M). Since $\hat{x}$ is a vector in the matroid polytope $\overline{\mathcal{F}}$ and $\mathbf{0} \leq \tilde{x} \leq \hat{x}$, the vector $\tilde{x}$ is also in $\overline{\mathcal{F}}$. This fact and the integrality of $\tilde{x}$ implies that $\tilde{X} \in \mathcal{F}$. We also have $c_i(\tilde{X}) = c_i^\top \tilde{x} \leq c_i^\top \hat{x} \leq B_i$ for all $i = 1, \dots, k$ since $\mathbf{0} \leq \tilde{x} \leq \hat{x}$. Hence, $\tilde{X}$ is a feasible solution to (kBM$^\natural$M).

We next show the inequality $f(\tilde{X}) \geq \text{OPT} - 2k \max_{v \in N} f(\{v\})$. We use the following property of the concave closure $\overline{f}$ of an M$^\natural$-concave function f.

Lemma A.3 ([28, 31, 37]).
(i) *Let $x, y \in \overline{\mathcal{F}}$ be vectors with $x \leq y$, $v \in N$, and $\alpha \in \mathbb{R}_+$ be a real number such that $y + \alpha\chi_v \in \overline{\mathcal{F}}$. Then, it holds that*

$$x + \alpha\chi_v \in \overline{\mathcal{F}}, \quad \overline{f}(x + \alpha\chi_v) - \overline{f}(x) \geq \overline{f}(y + \alpha\chi_v) - \overline{f}(y).$$

(ii) *For every $v \in N$ and $\alpha \in [0, 1]$, it holds that*

$$\overline{f}(\alpha\chi_v) - \overline{f}(\mathbf{0}) = \alpha\{f(\{v\}) - f(\emptyset)\}.$$

Let $u \in N$ be any element with $0 < \hat{x}(u) < 1$, and $\hat{x}' \in [0,1]^N$ be a vector given by $\hat{x}' = \hat{x} - \hat{x}(u)\chi_u$. It holds that

$$\begin{aligned}\overline{f}(\hat{x}) &\leq \overline{f}(\hat{x}') + \overline{f}(\hat{x}(u)\chi_u) - \overline{f}(\mathbf{0}) \\ &= \overline{f}(\hat{x}') + \hat{x}(u) f(\{u\}) \leq \overline{f}(\hat{x}') + \max_{v \in N} f(\{v\}),\end{aligned}$$

where the inequality is by Lemma A.3 (i) and the first equality is by Lemma A.3 (ii). By repeated application of this argument, we obtain the inequality

$$\textsc{opt} \leq \overline{f}(\hat{x}) \leq \overline{f}(\tilde{x}) + 2k \max_{v \in N} f(\{v\}) = f(\tilde{X}) + 2k \max_{v \in N} f(\{v\});$$

recall that there exist at most $2k$ non-integral components in $\hat{x}$ by Lemma 3.3.

A.5 Proof of Lemma 1.1 (i)

We show that for every $x \in \overline{\mathcal{F}}$ and $\delta > 0$, we can compute $p \in \mathbb{Q}^N$ and $\beta \in \mathbb{Q}$ satisfying

$$\overline{f}(y) - \overline{f}(x) \leq p^\top (y - x) + \delta \ (\forall y \in \overline{\mathcal{F}}), \quad \overline{f}(x) \leq \beta \leq \overline{f}(x) + \delta \tag{20}$$

in time polynomial in n and in $\log \max_{X \in \mathcal{F}} |f(X)|$.

We define a function $\overline{g} : \mathbb{R}^N \to \mathbb{R}$ as in Section 3, i.e.,

$$\overline{g}(p) = \inf\{p^\top y - \overline{f}(y) \mid y \in \overline{\mathcal{F}}\} \quad (p \in \mathbb{R}^N).$$

Lemma A.4 ([37, Lemma 3.3]). *Let $p \in \mathbb{R}^N$ be a vector satisfying $||p - p^*||_\infty \leq \delta/n$ for some $p^* \in \arg\max\{\overline{g}(q) - q^\top x \mid q \in \mathbb{R}^N\}$. Then, the vector p and the value $\beta = -\overline{g}(p) + p^\top x$ satisfy the inequalities in* (20).

Recall that $p^* \in \arg\max\{\overline{g}(q) - q^\top x \mid q \in \mathbb{R}^N\}$ holds if and only if p^* is a subgradient of $\overline{f}$ at $x \in \overline{\mathcal{F}}$, and that there exists such p^* satisfying $|p^*(v)| \leq 2n \max_{X \in \mathcal{F}} |f(X)|$ for all $v \in N$ (see Lemma 3.1).

We can compute a vector p in Lemma A.4 by finding a maximizer of the function $\overline{g}(q) - q^\top x$ among vectors q such that each component of q is an integer multiple of $\delta' = \delta/n^2$. We consider a function $h : \mathbb{Z}^N \to \mathbb{R}$ defined by

$$h(q) = \overline{g}(\delta' q) - \delta' q^\top x \quad (q \in \mathbb{Z}^N).$$

Lemma A.5 ([37, Theorem 3.5]). *Let $p' \in \mathbb{Z}^N$ be a maximizer of h under the constraint that $|\delta' p'(v)| \leq 2n \max_{X \in \mathcal{F}} |f(X)|$ $(v \in N)$. Then, there exists a maximizer $p^* \in \arg\max\{\overline{g}(q) - q^\top x \mid q \in \mathbb{R}^N\}$ such that*

$$||p^* - \delta' p'||_\infty \leq n\delta' = \frac{\delta}{n}.$$

By Lemmas A.4 and A.5, it suffices to solve the following problem:

$$\text{Maximize } h(q) \text{ subject to } |q(v)| \leq \frac{1}{\delta'} \cdot 2n \max_{X \in \mathcal{F}} |f(X)| \ (v \in N),$$

which can be solved exactly in time polynomial in n, in $\log \max_{X \in \mathcal{F}} |f(X)|$, and in $\log(1/\delta')$ by using the fact that the function $\overline{g}$ is an L-concave function [27, 28]. This concludes the proof of Lemma 1.1 (i).

A.6 Proof of Lemma 3.6

We show that for a fixed $\varepsilon > 0$, a feasible solution $x \in [0,1]^N$ of (CR) with $\overline{f}(x) \geq (1-\varepsilon)\overline{\text{OPT}}$ can be computed in polynomial time.

For $\beta \in \mathbb{R}$, we define the set

$$L(\beta) = \{(y, \alpha) \in [0,1]^N \times \mathbb{R} \mid y \in S,\ \beta \leq \alpha \leq \overline{f}(y)\}.$$

Recall that $S \subseteq [0,1]^N$ denotes the set of feasible solutions to (CR). We show that the nonemptyness of $L(\beta)$ can be checked approximately in polynomial time. We denote by $\overline{\text{OPT}}$ the optimal value of (CR).

Lemma A.6. *There exists a polynomial-time algorithm which, for every $\beta \in \mathbb{R}$ and $\varepsilon' > 0$, either asserts $\beta > \overline{\text{OPT}} - \varepsilon'$ or finds a feasible solution $x \in [0,1]^N$ of* (CR) *such that $\beta \leq \overline{f}(x) + \varepsilon'$.*

The proof of Lemma A.6 is given below. By combining this property and binary search with respect to β, we can compute a feasible solution $x \in [0,1]^N$ of (CR) with $\overline{f}(x) \geq (1-\varepsilon)\overline{\text{OPT}}$ in polynomial time, as follows.

In each iteration we maintain an interval $[\underline{\beta}, \overline{\beta}]$ and a feasible solution x_* of (CR) such that

$$\underline{\beta} \leq \overline{f}(x_*) + \frac{\varepsilon}{3} \cdot \max_{v \in N} f(\{v\}), \quad \overline{\beta} \geq \overline{\text{OPT}} - \frac{\varepsilon}{3} \cdot \max_{v \in N} f(\{v\}).$$

Note that $\max_{v \in N} f(\{v\})$ (> 0) is a non-zero lower bound of $\overline{\text{OPT}}$ which can be computed easily. Initially, we set $\underline{\beta} = 0$, $\overline{\beta} = \sum_{v \in N} f(\{v\})$, and $x_* = \mathbf{0}$; the value $\sum_{v \in N} f(\{v\})$ is an upper bound of the function value of f, and therefore it is also an upper bound of $\overline{\text{OPT}}$.

In each iteration, we use Lemma A.6 with $\beta = (\underline{\beta} + \overline{\beta})/2$ and $\varepsilon' = \varepsilon/3 \cdot \max_{v \in N} f(\{v\})$. If $\beta > \overline{\text{OPT}} - \varepsilon'$ holds, then we update $\overline{\beta} = \beta$, and proceed to the next iteration. If we find a feasible solution $x \in [0,1]^N$ of (CR) such that $\beta \leq \overline{f}(x) + \varepsilon'$, then we update $\underline{\beta} = \beta$, $x_* = x$, and proceed to the next iteration.

Suppose that $\overline{\beta} - \underline{\beta} \leq \varepsilon'$ holds in some iteration. Then, it holds that

$$\begin{aligned}\overline{f}(x_*) &\geq \underline{\beta} - \varepsilon' \geq \overline{\beta} - 2\varepsilon' \geq \overline{\text{OPT}} - 3\varepsilon' \\ &= \overline{\text{OPT}} - \varepsilon \cdot \max_{v \in N} f(\{v\}) \geq (1-\varepsilon)\overline{\text{OPT}}.\end{aligned}$$

Hence, the current x_* is a desired feasible solution of (CR). This implies that the number of iterations required by binary search is

$$\mathrm{O}\left(\log \frac{\sum_{v \in N} f(\{v\})}{\varepsilon/3 \cdot \max_{v \in N} f(\{v\})}\right) = \mathrm{O}\left(\log \frac{3n}{\varepsilon}\right).$$

which is polynomial in the input size.

Below we give a proof of Lemma A.6, which is done by using the ellipsoid method of Grötschel et al. [17], in a similar way as in Shioura [37]. By the results in [17, Chapter 3], it suffices to prove that the following oracle for the set $L(\beta)$ is available:

for every $(y, \alpha) \in [0,1]^N \times \mathbb{R}$ and $\delta > 0$, either asserts that y is a feasible solution to (CR) with $\beta \leq \alpha \leq \overline{f}(y) + \delta$, or outputs a vector $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}$ with $\|(s, \xi)\|_\infty = 1$ such that $s^\top(x - y) + \xi(\alpha' - \alpha) \leq \delta$ for all $(x, \alpha') \in L(\beta)$.

We construct such an oracle as follows.

Let $(y, \alpha) \in [0,1]^N \times \mathbb{R}$. We firstly check whether $y \in S$ or not, as in the case where f is an integer-valued function. If $y \notin S$, then we output a separating hyperplane of S (see Section 3).

Suppose that y is a feasible solution to (CR). If $\alpha < \beta$, then (y, α) is not in $L(\beta)$, and we output the vector $(\mathbf{0}, -1)$. If $\alpha \geq \beta$, then we compute an approximate value of $\overline{f}(y)$. By Lemma 1.1 (i) shown in Section A.5, we can compute in polynomial time $\eta \in \mathbb{Q}$ satisfying $\overline{f}(y) \leq \eta \leq \overline{f}(y) + \delta$. If $\eta \geq \alpha$, then we have $\alpha \leq \overline{f}(y) + \delta$, and therefore assert that y is a feasible solution to (CR) with $\beta \leq \alpha \leq \overline{f}(y) + \delta$. Otherwise (i.e., $\eta < \alpha$), the vector (y, α) is not in $L(\beta)$, and we compute an "approximate" subgradient of $\overline{f}$ at y. By Lemma 1.1 (i), we can compute in polynomial time $p \in \mathbb{Q}^N$ satisfying $\overline{f}(x) - \overline{f}(y) \leq p^\top(x - y) + \delta$ $(\forall x \in \overline{\mathcal{F}})$. It holds that $\overline{f}(y) \leq \eta < \alpha$ and $\alpha' \leq \overline{f}(x)$ for all $(x, \alpha') \in L(\beta)$. Hence,

$$\alpha' - \alpha < \overline{f}(x) - \overline{f}(y) \leq p^\top(x - y) + \delta$$

holds for all $(x, \alpha') \in L(\beta)$. Therefore, we output the vector $(1/\|(-p, 1)\|_\infty)(-p, 1)$. This concludes the proof of Lemma A.6.

A.7 Proof of Theorem 1.2

In §2, we have given a proof of Theorem 1.2 for the case where f is an integer-valued function. We here consider the general case where f is not necessarily an integer-valued function, and show a slightly weaker statement:

Lemma A.7. *For every fixed $\varepsilon > 0$, a feasible solution $\tilde{X} \in 2^N$ of (kBM$^\natural$M) satisfying*

$$f(\tilde{X}) \geq (1 - \varepsilon)\text{OPT} - 2k \max_{v \in N} f(\{v\}) \tag{21}$$

can be computed in polynomial time.

Although the bound in (21) is slightly weaker than the bound $\text{OPT} - 2k \max_{v \in N} f(\{v\})$ in Theorem 1.2, it is sufficient to obtain a PTAS for (kBM$^\natural$M).

Let $S, T \in 2^N$ be disjoint sets. We denote by (CR$[S, T]$) the problem (CR) with the additional constraints that $x(v) = 0$ for $v \in S$ and $x(v) = 1$ for $v \in T$. Similarly, we denote by (P$[S, T]$) the problem (kBM$^\natural$M) with the additional constraints that $X \cap S = \emptyset$ and $T \subseteq X$. That is, (P$[S, T]$) is the problem formulated as

$$\text{Maximize } f_T^{N \setminus S}(X) \text{ subject to } X \in \mathcal{F}_T^{N \setminus S}, \ \sum_{v \in X \cup T} c_i(v) \leq B_i \ (1 \leq i \leq k);$$

recall the definitions of $\mathcal{F}_T^{N\setminus S}$ and $f_T^{N\setminus S}$ in (4) and in (5), respectively. Hence, (P$[S,T]$) is an instance (kBM$^\natural$M). Note that (CR$[S,T]$) coincides with the continuous relaxation of (P$[S,T]$), which follows from the fact that $\mathcal{F}$ is the family of matroid independent sets and f is an M$^\natural$-concave function. This observation and Lemma 3.6 shown in Section A.6 imply that for every $\varepsilon > 0$ we can compute $(1-\varepsilon)$-approximate solution of (CR$[S,T]$) in polynomial time.

Below we present an algorithm which finds a pair of disjoint sets $F_0, F_1 \subseteq N$ with $|F_0 \cup F_1| \geq n - 2k$ such that there exists a $(1-\varepsilon)$-approximate solution x of (CR) with $x(v) = 0$ for $v \in F_0$ and $x(v) = 1$ for $v \in F_1$. Once we obtain such sets, we can show in the same way as Lemma 3.5 that the set $\tilde{X} = F_1$ is a feasible solution to (kBM$^\natural$M) satisfying the inequality (21).

The algorithm maintains two sets F_0 and F_1 satisfying the condition that

$$\text{the optimal value of } (\text{CR}[F_0, F_1]) \geq (1 - \frac{|F_0 \cup F_1|\varepsilon}{n})\overline{\text{OPT}}. \tag{22}$$

Initially, we set $F_0 = \emptyset$ and $F_1 = \emptyset$, and in the following iterations an element in $N \setminus (F_0 \cup F_1)$ is repeatedly added to either F_0 or F_1 until $|F_0 \cup F_1| \geq n - 2k$ holds.

In each iteration of the algorithm, we check whether an element $u \in N \setminus (F_0 \cup F_1)$ can be added to F_0 or F_1. For each $u \in N \setminus (F_0 \cup F_1)$, we compute values η_0^u and η_1^u such that

$$\overline{\text{OPT}}_0^u \geq \eta_0^u \geq (1 - \frac{\varepsilon}{n})\overline{\text{OPT}}_0^u, \quad \overline{\text{OPT}}_1^u \geq \eta_1^u \geq (1 - \frac{\varepsilon}{n})\overline{\text{OPT}}_1^u,$$

where $\overline{\text{OPT}}_0^u$ (resp., $\overline{\text{OPT}}_1^u$) is the optimal value of (CR$[F_0 \cup \{u\}, F_1]$) (resp., (CR$[F_0, F_1 \cup \{u\}]$)). Let η_* be a value such that

$$\overline{\text{OPT}}_* \geq \eta_* \geq (1 - \frac{|F_0 \cup F_1|\varepsilon}{n})\overline{\text{OPT}}_*,$$

where $\overline{\text{OPT}}_*$ is the optimal value of (CR$[F_0, F_1]$). Note that such η_* is already computed in the previous iteration of the algorithm.

Suppose that $\eta_0^u \geq (1 - \frac{\varepsilon}{n})\eta_*$ holds for some $u \in N \setminus (F_0 \cup F_1)$. Then, we have

$$\begin{aligned}\overline{\text{OPT}}_0^u \geq \eta_0^u \geq (1 - \frac{\varepsilon}{n})\eta_* &\geq (1 - \frac{\varepsilon}{n})(1 - \frac{|F_0 \cup F_1|\varepsilon}{n})\overline{\text{OPT}} \\ &\geq (1 - \frac{|F_0 \cup F_1| + 1}{n} \cdot \varepsilon)\overline{\text{OPT}}.\end{aligned}$$

Hence, we add the element u to F_0, and proceed to the next iteration. Similarly, if $\eta_1^u \geq (1 - \frac{\varepsilon}{n})\eta_*$ holds for some $u \in N \setminus (F_0 \cup F_1)$, then we add u to F_1, and proceed to the next iteration.

Suppose that $\eta_0^u < (1 - \frac{\varepsilon}{n})\eta_*$ and $\eta_1^u < (1 - \frac{\varepsilon}{n})\eta_*$ hold for all $u \in N \setminus (F_0 \cup F_1)$. Then, we have

$$(1 - \frac{\varepsilon}{n})\overline{\text{OPT}}_b^u \leq \eta_b^u < (1 - \frac{\varepsilon}{n})\eta_* \leq (1 - \frac{\varepsilon}{n})\overline{\text{OPT}}_*,$$

i.e., $\overline{\text{OPT}}_b^u < \overline{\text{OPT}}_*$ holds for all $b = 0, 1$ and all $u \in N \setminus (F_0 \cup F_1)$. This means that any optimal solution of the problem (CR$[F_0, F_1]$) has no integral component. On the other hand, the problem (CR$[F_0, F_1]$) has $n' = n - |F_0 \cup F_1|$ free variables, and Lemma 3.3 applied to (CR$[F_0, F_1]$) implies that there exists an optimal solution of (CR$[F_0, F_1]$) which has at least $(n' - 2k)$ integral components. Hence, we must have $n' \leq 2k$, i.e., $|F_0 \cup F_1| \geq n - 2k$ holds. Therefore, we can stop the algorithm in this case. This concludes the proof of Lemma A.7.

A.8 M$^\natural$-concave Intersection Problem without Budget Constraint

We show that the M$^\natural$-concave intersection problem without budget constraint can be solved in polynomial time by a "combinatorial" algorithm in the sense of Megiddo [24]. A "combinatorial" algorithm is an algorithm which applies only comparison and addition operations to input numbers such as function values; this means, in particular, the multiplication of input numbers is not allowed. To show this, we use a reduction to the valuated matroid intersection problem discussed by Murota [26].

M-concave Function We explain the concept of M-concave function, which is used in the objective function of the valuated matroid intersection problem.

Let $\mathcal{B} \subseteq 2^N$ be the family of bases in a matroid; such $\mathcal{B}$ can be characterized by the following property (see, e.g., [28]):

$$\forall X, Y \in \mathcal{B},\ \forall u \in X \setminus Y,\ \exists v \in Y \setminus X\colon X - u + v \in \mathcal{B},\ Y + u - v \in \mathcal{B}.$$

Recall that $|X| = |Y|$ for every $X, Y \in \mathcal{B}$.

We consider a function $g : \mathcal{B} \to \mathbb{R}$ defined on a base family $\mathcal{B}$, which is called an *M-concave function* [25, 27] if it satisfies the following property:

(M-EXC) $\forall X, Y \in \mathcal{B},\ \forall u \in X \setminus Y,\ \exists v \in Y \setminus X$:

$$X - u + v \in \mathcal{B},\ Y + u - v \in \mathcal{B}, \qquad g(X) + g(Y) \leq g(X - u + v) + g(Y + u - v).$$

The concept of M-concave function is originally introduced for functions defined on integer lattice points, and is deeply related to that of M$^\natural$-concave function. For set functions, M-concavity is equivalent to the concept of valuated matroid by Dress and Wenzel [9].

For every M$^\natural$-concave function $f : \mathcal{F} \to \mathbb{R}$, we can construct an M-concave function $g : \mathcal{B} \to \mathbb{R}$ which is essentially equivalent to f. Let $r = \max\{|X| \mid X \in \mathcal{F}\}$. Also, let $u_1, u_2, \ldots, u_r$ be elements not in N, and put $\tilde{N} = N \cup \{u_1, u_2, \ldots, u_r\}$. Define $\tilde{\mathcal{B}} \subseteq 2^{\tilde{N}}$ and a function $g : \tilde{\mathcal{B}} \to \mathbb{R}$ by

$$\tilde{\mathcal{B}} = \{\tilde{X} \subseteq \tilde{N} \mid |\tilde{X}| = r,\ \tilde{X} \cap N \in \mathcal{F}\}, \qquad g(\tilde{X}) = f(\tilde{X} \cap N) \quad (\tilde{X} \in \tilde{\mathcal{B}}).$$

We show that g is indeed an M-concave function. In the proof below we use the following property of M$^\natural$-concave functions.

Lemma A.8 ([30]). *Let $f : \mathcal{F} \to \mathbb{R}$ be an M$^\natural$-concave function and $X, Y \in \mathcal{F}$.*
(i) *If $|X| \leq |Y|$, then for every $u \in X \setminus Y$ there exists some $v \in Y \setminus X$ such that*

$$X - u + v \in \mathcal{F},\ Y + u - v \in \mathcal{F},\ f(X) + f(Y) \leq f(X - u + v) + f(Y + u - v).$$

(ii) *If $|X| < |Y|$, then there exists some $v \in Y \setminus X$ such that*

$$X + v \in \mathcal{F},\ Y - v \in \mathcal{F},\ f(X) + f(Y) \leq f(X + v) + f(Y - v).$$

Proof (M-concavity for g). Let $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{B}}$ and $u \in \tilde{X} \setminus \tilde{Y}$. We show that there exists some $v \in \tilde{Y} \setminus \tilde{X}$ such that

$$\tilde{X} - u + v \in \tilde{\mathcal{B}},\ \tilde{Y} + u - v \in \tilde{\mathcal{B}},\ g(\tilde{X}) + g(\tilde{Y}) \leq g(\tilde{X} - u + v) + g(\tilde{Y} + u - v). \quad (23)$$

Put $X = \tilde{X} \cap N$ and $Y = \tilde{Y} \cap N$. By definition, we have $X, Y \in \mathcal{F}$, $g(\tilde{X}) = f(X)$, and $g(\tilde{Y}) = f(Y)$.

[Case 1: $u \in N$] We have $u \in X \setminus Y$. By (M$^\natural$-EXC), we have either (a) or (b) (or both) holds:

(a) $X - u, Y + u \in \mathcal{F}$ and $f(X) + f(Y) \leq f(X - u) + f(Y + u)$,
(b) $\exists v \in Y \setminus X$: $X - u + v$, $Y + u - v \in \mathcal{F}$ and $f(X) + f(Y) \leq f(X - u + v) + f(Y + u - v)$.

By Lemma A.8 (i), the statement (b) always holds whenever $|X| \leq |Y|$.

Suppose that (a) occurs. Then, we may assume $|X| > |Y|$. Since $|\tilde{X}| = |\tilde{Y}|$, there exists some $v = u_h \in (\tilde{Y} \setminus \tilde{X}) \setminus N$. We have $\tilde{X} - u + v, \tilde{Y} + u - v \in \tilde{\mathcal{B}}$ since $(\tilde{X} - u + v) \cap N = X - u \in \mathcal{F}$ and $(\tilde{Y} + u - v) \cap N = Y + u \in \mathcal{F}$. Moreover, it holds that

$$\begin{aligned} g(\tilde{X}) + g(\tilde{Y}) = f(X) + f(Y) &\leq f(X - u) + f(Y + u) \\ &= g(\tilde{X} - u + v) + g(\tilde{Y} + u - v). \end{aligned}$$

Hence, (23) holds.

We then suppose that (b) occurs. Then, the element $v \in Y \setminus X$ in (b) satisfies $v \in \tilde{Y} \setminus \tilde{X}$, $\tilde{X} - u + v, \tilde{Y} + u - v \in \tilde{\mathcal{B}}$, and

$$\begin{aligned} g(\tilde{X}) + g(\tilde{Y}) = f(X) + f(Y) &\leq f(X - u + v) + f(Y + u - v) \\ &= g(\tilde{X} - u + v) + g(\tilde{Y} + u - v). \end{aligned}$$

Hence, (23) holds.

[Case 2: $u \in \tilde{N} \setminus N$] We denote $U = \{u_1, u_2, \ldots, u_r\}$. Suppose that there exists some $v \in (\tilde{Y} \setminus \tilde{X}) \cap U$. Then, we have $\tilde{X} - u + v, \tilde{Y} + u - v \in \tilde{\mathcal{B}}$ and

$$g(\tilde{X}) + g(\tilde{Y}) = f(X) + f(Y) = g(\tilde{X} - u + v) + g(\tilde{Y} + u - v),$$

i.e., (23) holds.

We then consider the case where $(\tilde{Y} \setminus \tilde{X}) \cap U = \emptyset$. Then, we have $\tilde{Y} \cap U \subseteq (\tilde{X} \cap U) \setminus \{u\}$, implying that $|\tilde{Y} \cap U| < |\tilde{X} \cap U|$. Since $|\tilde{X}| = |\tilde{Y}|$, it holds that $|X| = |\tilde{X}| - |\tilde{Y} \cap U| < |\tilde{Y}| - |\tilde{X} \cap U| = |Y|$. By Lemma A.8 (ii), there exists some

$v \in Y \setminus X$ such that $X + v, Y - v \in \mathcal{F}$ and $f(X) + f(Y) \leq f(X + v) + f(Y - v)$. This implies that $v \in \tilde{Y} \setminus \tilde{X}$, $\tilde{X} - u + v, \tilde{Y} + u - v \in \tilde{\mathcal{B}}$, and

$$g(\tilde{X}) + g(\tilde{Y}) = f(X) + f(Y) \leq f(X + v) + f(Y - v) \\ = g(\tilde{X} - u + v) + g(\tilde{Y} + u - v),$$

i.e., (23) holds. □

Note that the integer r in the construction of g can be computed in polynomial time by using an algorithm for the (unweighted) matroid intersection problem.

Reduction to Valuated Matroid Intersection Problem The *valuated matroid intersection problem* considered in [26] is formulated as follows:

$$\text{Maximize } g_1(X) + g_2(X) \text{ subject to } X \in \mathcal{B}_1 \cap \mathcal{B}_2,$$

where $g_i : \mathcal{B}_i \to \mathbb{R}$ $(i = 1, 2)$ are M-concave functions defined on matroid base families $\mathcal{B}_i$ satisfying $\mathcal{B}_1 \cap \mathcal{B}_2 \neq \emptyset$. It is shown in [26] that this problem can be solved in polynomial time by an augmenting-type "combinatorial" algorithm. We show that the maximization of the sum of two M$^\natural$-concave functions can be reduced to the valuated matroid intersection problem.

Consider the maximization of the sum of two M$^\natural$-concave functions $f_i : \mathcal{F}_i \to \mathbb{R}$ $(i = 1, 2)$ defined on the families of matroid independent sets $\mathcal{F}_i$. For each M$^\natural$-concave function f_i, we consider an M-concave functions $g_i : \tilde{\mathcal{B}}_i \to \mathbb{R}$ defined as in the previous section, where $\tilde{\mathcal{B}}_i \subseteq 2^{\tilde{N}}$. We have $\tilde{\mathcal{B}}_1 \cap \tilde{\mathcal{B}}_2 \neq \emptyset$ since $\{u_1, \ldots, u_r\} \in \tilde{\mathcal{B}}_i$. Moreover, for a set $\tilde{X} \subseteq \tilde{N}$, we have $\tilde{X} \in \arg\max(g_1 + g_2)$ if and only if $|\tilde{X}| = r$ and $\tilde{X} \cap N \in \arg\max(f_1 + f_2)$. This shows that the maximization of the sum of two M$^\natural$-concave functions can be reduced to the valuated intersection problem, and hence can be solved by a "combinatorial" algorithm in polynomial time.

A.9 Proof of Theorem 1.5

In this section we give a more detailed proof of Theorem 1.5 for (1BM$^\natural$I), showing that a set $\tilde{X} \in \mathcal{F}_1 \cap \mathcal{F}_2$ satisfying the condition

$$f_1(\tilde{X}) + f_2(\tilde{X}) \geq \text{OPT} - 2 \cdot \max_{v \in N}\{f_1(\{v\}) + f_2(\{v\})\}, \quad c(\tilde{X}) \leq B + \max_{v \in N} c(v) \quad (24)$$

can be computed in polynomial time. Recall the assumption that $\{v\} \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $f_1(\{v\}) + f_2(\{v\}) > 0$ hold for all $v \in N$.

We use the Lagrangian relaxation of (1BM$^\natural$I) in a similar way as in [2, 34]. Using a parameter $\lambda \in \mathbb{R}_+$ called *Lagrangian multiplier*, the Lagrangian relaxation is given by

(LR(λ)) Maximize $f_1(X) + f_2(X) + \lambda\{B - c(X)\}$ subject to $X \in \mathcal{F}_1 \cap \mathcal{F}_2$.

The problem (LR(λ)) is an instance of the M$^\natural$-concave intersection problem *without* budget constraint, which is essentially equivalent to the valuated matroid intersection problem discussed in [25] (see Section A.8). Therefore, the theorems and algorithms in [25] can be used to (LR(λ)) with slight modification. In particular, (LR(λ)) can be solved in polynomial time.

The value $\lambda = \lambda_*$ minimizing the optimal value of (LR(λ)) is called an *optimal Lagrangian multiplier*. We firstly compute an optimal Lagrangian multiplier λ_* and two optimal solutions X_*, Y_* of (LR(λ_*)) with $c(X_*) \leq B \leq c(Y_*)$. This can be done by Megiddo's parametric search technique (see [24]; see also [2, 34]). Recall that OPT denotes the optimal value of the problem (1BM$^\natural$I).

Lemma A.9. *Let $X_*, Y_* \in 2^N$ be optimal solutions of* (LR(λ_*)). *If $c(X_*) \leq B \leq c(Y_*)$, then $f(X_*) \leq$ OPT $\leq f(Y_*)$ holds.*

Proof. If $c(X_*) \leq B$ then X_* is a feasible solution of (1BM$^\natural$I). Hence, we have $f(X_*) \leq$ OPT. Since (LR(λ_*)) is a relaxation of (1BM$^\natural$I), it holds that $f(Y_*) + \lambda_*\{B - c(Y_*)\} \geq$ OPT, which, combined with $B \leq c(Y_*)$ and $\lambda_* \geq 0$, yields $f(Y_*) \geq$ OPT. □

This lemma implies that if $X_* = Y_*$, then we have $c(X_*) = B$ and $f(X_*) =$ OPT, and therefore $\tilde{X} = X_*$ satisfies the condition in Theorem 1.5. Otherwise (i.e., $X_* \neq Y_*$), we apply "patching" operations to X_* and Y_* to construct a better approximate solution.

The patching operations are done by using cycles in a weighted auxiliary graph constructed as follows. Given $X, Y \in \mathcal{F}_1 \cap \mathcal{F}_2$, we define an *auxiliary graph* $G_X^Y = (V, A)$ with arc weight $\omega : A \to \mathbb{R}$ as follows; $V = (X \setminus Y) \cup (Y \setminus X) \cup \{v_a, v_d\}$, where v_a and v_d are new elements not in N, and A and ω are given by

$$
\begin{aligned}
& A = E_1 \cup E_2 \cup A_1 \cup A_2 \cup D_1 \cup D_2, \\
& E_1 = \{(u, v) \mid u \in X \setminus Y,\ v \in Y \setminus X,\ X - u + v \in \mathcal{F}_1\}, \\
& \quad \omega(u, v) = f_1(X - u + v) - f_1(X) + \lambda_*\{c(u) - c(v)\} \quad ((u, v) \in E_1), \\
& E_2 = \{(v, u) \mid v \in Y \setminus X,\ u \in X \setminus Y,\ X + v - u \in \mathcal{F}_2\}, \\
& \quad \omega(v, u) = f_2(X + v - u) - f_2(X) \quad ((v, u) \in E_2), \\
& A_1 = \{(v_a, v) \mid v \in Y \setminus X,\ X + v \in \mathcal{F}_1\}, \\
& \quad \omega(v_a, v) = f_1(X + v) - f_1(X) - \lambda_* c(v) \quad ((v_a, v) \in A_1), \\
& A_2 = \{(v, v_a) \mid v \in Y \setminus X,\ X + v \in \mathcal{F}_2\}, \\
& \quad \omega(v, v_a) = f_2(X + v) - f_2(X) \quad ((v, v_a) \in A_2), \\
& D_1 = \{(u, v_d) \mid u \in X \setminus Y,\ X - u \in \mathcal{F}_1\}, \\
& \quad \omega(u, v_d) = f_1(X - u) - f_1(X) + \lambda_* c(u) \quad ((u, v_d) \in D_1), \\
& D_2 = \{(v_d, u) \mid u \in X \setminus Y,\ X - u \in \mathcal{F}_2\}, \\
& \quad \omega(v_d, u) = f_2(X - u) - f_2(X) \quad ((v_d, u) \in D_2).
\end{aligned}
$$

The auxiliary graph defined here is a variant of the auxiliary graph for the valuated matroid intersection problem used in [25]. Hence, properties of the auxiliary graph for the valuated matroid intersection problem can be used for the auxiliary graph G_X^Y with some appropriate modification.

A *cycle* in the graph G_X^Y is a directed closed path which visits each vertex at most once. We note that in every cycle in G_X^Y, arcs in $E_1 \cup A_1 \cup D_1$ and arcs in $E_2 \cup A_2 \cup D_2$ appear alternately, and therefore every cycle contains an even number of arcs. We call a cycle in G_X^Y *admissible* if the cycle visits at most one of v_a and v_d.

For an admissible cycle C in G_X^Y, we define a set $X \oplus C$ $(\subseteq N)$ by

$$X \oplus C = X \setminus \{u \in X \setminus Y \mid (u, v) \in C \cap E_1 \text{ or } (u, v_d) \in C \cap D_1\} \\ \cup \{v \in Y \setminus X \mid (u, v) \in C \cap E_1 \text{ or } (v_a, v) \in C \cap A_1\}.$$

It is easy to see that

- if C visits neither of v_a and v_d, then $|X \oplus C| = |X|$,
- if C visits v_a but not v_d, then $|X \oplus C| = |X| + 1$,
- if C visits v_d but not v_a, then $|X \oplus C| = |X| - 1$.

To the end of this section, we denote

$$f_{\lambda_*}(X) = f_1(X) + f_2(X) + \lambda_*\{B - c(X)\} \qquad (X \in 2^N).$$

Lemma A.10 (cf. [25]). *Let $X, Y \in \mathcal{F}_1 \cap \mathcal{F}_2$.*
(i) *Let C be a maximum-weight admissible cycle in G_X^Y with the minimum number of arcs. Then, $X \oplus C \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $f_{\lambda_*}(X \oplus C) = f_{\lambda_*}(X) + \omega(C)$, where $\omega(C)$ denotes the total weight of the cycle C.*
(ii) *If X is an optimal solution of* (LR(λ_*))*, then there exists no positive-weight admissible cycle in G_X^Y.*
(iii) *If Y is an optimal solution of* (LR(λ_*)) *and X is not optimal, then there exists a positive-weight admissible cycle in G_X^Y.*

Lemma A.11. *The weight of a maximum-weight admissible cycle in $G_{X_*}^{Y_*}$ is zero.*

Proof. By Lemma A.10 (ii), there is no admissible cycle in $G_{X_*}^{Y_*}$ which has positive weight. We consider a slight purturbation of the objective function in (LR(λ_*)) so that Y_* is optimal but X_* is not. By applying Lemma A.10 (iii) to the purturbed problem, we can show that there is a positive-weight admissible cycle in the auxiliary graph $G_{X_*}^{Y_*}$ with respect to the perturbed problem. This implies the existence of an admissible cycle with zero weight in the original auxiliary graph $G_{X_*}^{Y_*}$, which is a maximum-weight admissible cycle. □

Our patching operation is based on the following property, which is immediate from Lemma A.10 (i) and Lemma A.11.

Lemma A.12. *Let C be a zero-weight admissible cycle in $G_{X_*}^{Y_*}$ with the minimum number of arcs. Then, $X_* \oplus C$ is an optimal solution of* (LR(λ_*)) *with $X_* \oplus C \neq X_*$.*

In the patching operation, we compute a zero-weight cycle C in $G_{X_*}^{Y_*}$ with the minimum number of arcs. This can be done by using a shortest-path algorithm since a zero-weight cycle is a maximum-weight cycle by Lemma A.11. Then, set $X' = X_* \oplus C$, which is an optimal solution of (LR(λ_*)) by Lemma A.12.

If $X' = Y_*$, then we apply an additional patching operation to be explained below. If $c(X') = B$, then X' satisfies the condition (24) and stop. If $c(X') < B$ then we replace X_* with X'; otherwise (i.e., $c(X') > B$), we replace Y_* with X'; in both cases, we repeat the same patching operations. It is noted that whenever the patching operation above is repeated, then the cardinality of the set $(X_* \setminus Y_*) \cup (Y_* \setminus X_*)$ decreases, and therefore the operation is repeated at most n times.

Finally, we explain the additional patching operation used in the case where $X_* \oplus C = Y_*$. In this case, the cycle C contains all vertices in the graph $G_{X_*}^{Y_*}$. Let $a_1, a_2, \ldots, a_{2h} \in A$ be a sequence of arcs in the cycle C, where $2h$ is the number of arcs in C. It may be assumed that $a_j \in E_1 \cup A_1 \cup D_1$ if j is odd and $a_j \in E_2 \cup A_2 \cup D_2$ if j is even. For $j = 1, 2, \ldots, h$, let $\alpha_j = \omega(a_{2j-1}) + \omega(a_{2j})$. Since C is a zero-weight cycle, we have $\sum_{j=1}^{h} \alpha_j = 0$.

Lemma A.13 (Gasoline Lemma (cf. [23])). *Let $\alpha_1, \alpha_2, \ldots, \alpha_h \in \mathbb{R}$ be a sequence of real numbers satisfying $\sum_{j=1}^{h} \alpha_j = 0$. Then, there exists some $t \in \{1, 2, \ldots, h\}$ such that*

$$\sum_{j=t}^{t+i} \alpha_{j (\mathrm{mod}\, h)} \geq 0 \qquad (i = 0, 1, \ldots, h-1),$$

where $\alpha_0 = \alpha_h$. □

By Gasoline Lemma, we may assume that $\sum_{j=1}^{i} \alpha_j \geq 0$ for all $i = 1, 2, \ldots, h$. For $j = 1, 2, \ldots, h$, we define $\eta_j \in \mathbb{R}$ by

$$\eta_j = \begin{cases} c(v) - c(u) & (a_{2j-1} = (u, v) \in E_1), \\ c(v) & (a_{2j-1} = (v_a, v) \in A_1), \\ -c(u) & (a_{2j-1} = (u, v_d) \in D_1). \end{cases}$$

Then, $c(Y_*) = c(X_*) + \sum_{j=1}^{h} \eta_j$ holds. Let $t \in \{1, 2, \ldots, h\}$ be the minimum integer such that $c(X_*) + \sum_{j=1}^{t} \eta_j > B$. Since $c(X_*) < B$, we have $t \geq 1$. In addition, the choice of t implies that $c(X_*) + \sum_{j=1}^{t-1} \eta_j \leq B$.

In the following, we assume that $C \subseteq E_1 \cup E_2$ for simplicity of the description, since the remaining cases can be shown similarly. We consider the arc set $C' = \{a_1, a_2, \ldots, a_{2t-1}, a_{2t}\}$. For $j = 1, 2, \ldots, h$, denote $a_{2j-1} = (u_j, v_j)$. We define a set $\tilde{X} \subseteq N$ by

$$\begin{aligned} \tilde{X} = X_* \setminus \{u \in X_* \mid (u, v) \in C' \cap E_1 \text{ or } u = u_{t+1}\} \\ \cup \{v \in N \setminus X_* \mid (u, v) \in C' \cap E_1\}. \end{aligned}$$

We show that the set $\tilde{X}$ satisfies $\tilde{X} \in \mathcal{F}_1 \cap \mathcal{F}_2$ and the condition (24).

We have

$$\begin{aligned} c(\tilde{X}) = c(X_*) + \sum_{j=1}^{t} \eta_j - c(u_{t+1}) &\leq \{c(X_*) + \sum_{j=1}^{t-1} \eta_j\} + \eta_t \\ &\leq B + \eta_t \leq B + \max_{v \in N} c(v). \end{aligned}$$

Hence, $\tilde{X}$ satisfies the second inequality in (24).

We also have

$$
\begin{aligned}
\text{OPT} &\le f_1(X_*) + f_2(X_*) + \lambda_*\{B - c(X_*)\} \\
&\le f_1(X_*) + f_2(X_*) + \lambda_*\{B - c(X_*)\} + \sum_{j=1}^{t} \alpha_j \\
&= \left[f_1(X_*) + \sum_{j=1}^{t} \{f_1(X_* - u_j + v_j) - f_1(X_*)\} \right] \\
&\quad + \left[f_2(X_*) + \sum_{j=1}^{t} \{f_1(X_* - u_{j+1} + v_j) - f_1(X_*)\} \right] \\
&\quad + \lambda_* \left[\{B - c(X_*)\} - \sum_{j=1}^{t} \eta_j \right] \\
&< \left[f_1(X_*) + \sum_{j=1}^{t} \{f_1(X_* - u_j + v_j) - f_1(X_*)\} \right] \\
&\quad + \left[f_2(X_*) + \sum_{j=1}^{t} \{f_1(X_* - u_{j+1} + v_j) - f_1(X_*)\} \right], \qquad (25)
\end{aligned}
$$

where the last inequality is by the choice of t.

We define $\tilde{X}_1, \tilde{X}_2 \subseteq N$ by

$$
\begin{aligned}
\tilde{X}_1 &= X_* \setminus \{u_1, \ldots, u_t\} \cup \{v_1, \ldots, v_t\} \ (= \tilde{X} \cup \{u_{t+1}\}), \\
\tilde{X}_2 &= X_* \setminus \{u_2, u_3, \ldots, u_t, u_{t+1}\} \cup \{v_1, \ldots, v_t\} \ (= \tilde{X} \cup \{u_1\}).
\end{aligned}
$$

By using the fact that C' is a subpath of a zero-weight admissible cycle with the smallest number of arcs, we can show the following properties by using a similar proof technique as in [25]:

$$
\begin{aligned}
&\tilde{X}_1 \in \mathcal{F}_1, \ \tilde{X}_2 \in \mathcal{F}_2, \\
&f_1(\tilde{X}_1) = f_1(X_*) + \sum_{j=1}^{t} \{f_1(X_* - u_j + v_j) - f_1(X_*)\}, \qquad (26) \\
&f_2(\tilde{X}_2) = f_2(X_*) + \sum_{j=1}^{t} \{f_1(X_* - u_{j+1} + v_j) - f_1(X_*)\}. \qquad (27)
\end{aligned}
$$

Hence, we obtain $\tilde{X} \in \mathcal{F}_1 \cap \mathcal{F}_2$ since $\tilde{X}$ is a common subset of $\tilde{X}_1$ and $\tilde{X}_2$. From (25), (26), and (27) follows

$$
f_1(\tilde{X}_1) + f_2(\tilde{X}_2) \ge \text{OPT}.
$$

By the submodularity of f_1 and f_2 in the sense of (3), it holds that

$$\begin{aligned} f_1(\tilde{X}) + f_2(\tilde{X}) &\geq f_1(\tilde{X}_1) - \{f_1(\tilde{X}_1) - f_1(\tilde{X})\} + f_2(\tilde{X}_2) - \{f_2(\tilde{X}_2) - f_2(\tilde{X})\} \\ &\geq f_1(\tilde{X}_1) - \{f_1(\{u_{t+1}\}) - f_1(\emptyset)\} + f_2(\tilde{X}_2) - \{f_2(\{u_1\}) - f_2(\emptyset)\} \\ &\geq \text{OPT} - 2 \cdot \max_{v \in N} \{f_1(\{v\}) + f_2(\{v\})\}. \end{aligned}$$

Hence, the set $\tilde{X}$ satisfies the first inequality in (24). This concludes the proof of Theorem 1.5.

A.10 Extension of Our Results

Our PTAS for (kBM$^\natural$M) can be extended for a more general problem with M$^\natural$-concave objective functions defined on integral polymatroids. Let $P \subseteq \mathbb{Z}_+^N$ be an integral polymatroid, i.e., P is given as

$$P = \{x \in \mathbb{Z}_+^N \mid x(S) \leq \rho(S) \ (S \in 2^N)\}$$

with an integer-valued monotone submodular function $\rho : 2^N \to \mathbb{Z}_+$ satisfying $\rho(\emptyset) = 0$.

We consider a function $h : P \to \mathbb{R}$, which is called an *M$^\natural$-concave function* if it satisfies the following property:

$\forall x, y \in P$, $\forall u \in \text{supp}^+(x - y)$, either (i) or (ii) (or both) holds:
(i) $x - \chi_u, y + \chi_u \in P$ and

$$h(x) + h(y) \leq h(x - \chi_u) + h(y + \chi_u),$$

(ii) $\exists v \in \text{supp}^-(x - y)$: $x - \chi_u + \chi_v, y + \chi_u - \chi_v \in P$ and

$$h(x) + h(y) \leq h(x - \chi_u + \chi_v) + h(y + \chi_u - \chi_v),$$

where $\chi_u \in \{0,1\}^N$ is the characteristic vector of $u \in N$, and for vectors $x, y \in \mathbb{R}^N$ we define $\text{supp}^+(x - y) = \{i \in N \mid x(i) > y(i)\}$ and $\text{supp}^-(x - y) = \{i \in N \mid x(i) < y(i)\}$. For an M$^\natural$-concave function $h : P \to \mathbb{R}$ with $h(\mathbf{0}) = 0$, we consider the following problem:

$$\text{Maximize} \ \ h(x) \quad \text{subject to} \ \ x \in P, \ c_i^\top x \leq B_i \ (1 \leq i \leq k),$$

where k is a positive integer given as a constant, $c_i \in \mathbb{R}_+^N$, and $B_i \in \mathbb{R}_+$. If $P \subseteq \{0,1\}^N$, then this problem is equivalent to (kBM$^\natural$M). Our PTAS for (kBM$^\natural$M) can be naturally extended to this problem by using the results in discrete convex analysis.