A Bound for the Number of Different Basic Solutions Generated by the Simplex Method

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Contents

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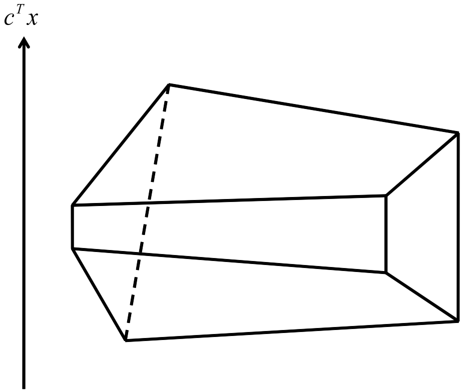
New Section

Simplex Method

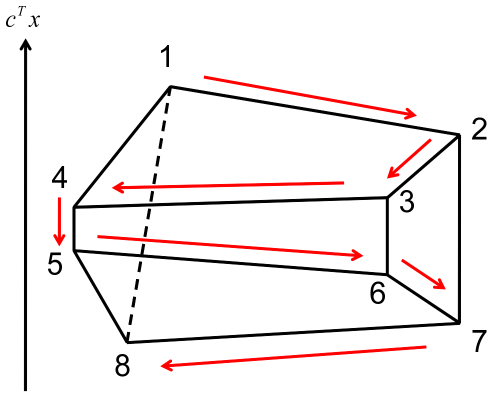
- The simplex method for LP was originally developed by Dantzig.
- In spite of the practical efficiency of the simplex method, we do not have any good bound for the number of iterations (the bound we knew was only the number of bases $\frac{n!}{m!(n-m)!}$).
- Klee and Minty showed that the simplex method needs an exponential number of iterations for an LP.

LP on a perturbed rectangular feasible region



Number of vertices (basic feasible solutions) is $\mathbf{2^n = 8}$.

Exponential number of iterations



Number of Iterations (or number of vertices generated) is also $\mathbf{2^n = 8}$.

LP on a rectangular feasible region

Suppose that

then

LP on a rectangular feasible region

Suppose that

- the feasible region is rectangular without perturbation (for example, all the components of vertices are **0** or **1** after scaling),

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LP on a rectangular feasible region

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- the feasible region is rectangular without perturbation (for example, all the components of vertices are **0** or **1** after scaling),
- we use Dantzig's rule (the most negative pivoting rule),

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LP on a rectangular feasible region

Suppose that

- the feasible region is rectangular without perturbation (for example, all the components of vertices are **0** or **1** after scaling),
- we use Dantzig's rule (the most negative pivoting rule),

then

- the number of vertices generated by the simplex method could not be exponential,

LP on a rectangular feasible region

Suppose that

- the feasible region is rectangular without perturbation (for example, all the components of vertices are $\mathbf{0}$ or $\mathbf{1}$ after scaling),
- we use Dantzig's rule (the most negative pivoting rule),

then

- the number of vertices generated by the simplex method could not be exponential,
- our result shows that the number is bounded by $\boldsymbol{nm \log m}$.

Standard form of LP

The standard form of LP is

$$\begin{array}{ll} \textbf{min} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \textbf{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \\ & (x_1, x_2, \cdots, x_n)^T \geq 0. \end{array}$$

or by using vectors and a matrix

$$\begin{array}{ll} \textbf{min} & c^T x, \\ \textbf{subject to} & Ax = b, \ x \geq 0. \end{array}$$

Main Result

The number of different basic feasible solutions (BFSs) generated by the simplex method with Dantzig's rule (the most negative pivoting rule) is bounded by

$$n\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil,$$

where δ and γ are the minimum and the maximum values of all the positive elements of primal BFSs and $\lceil a \rceil$ denotes the smallest integer greater than a.

When the primal problem is nondegenerate, it becomes a bound for the number of iterations.

The bound depends only on the constraints of LP.

Ye’s result for MDP

Our work is motivated by a recent research by Ye (2010). He shows that the simplex method is strongly polynomial for the Markov Decision Problem (MDP).
We utilize his analysis to general LPs and obtain the upper bound.
Our results include his strong polynomiality.

LP with Unimodular Matrix

When we apply our result to an LP where a constraint matrix is totally unimodular and a constant vector $\boldsymbol{b}$ is integral, the number of different solutions generated by the simplex method is at most

$$n\lceil m\|\boldsymbol{b}\|_1 \log(m\|\boldsymbol{b}\|_1)\rceil.$$

New Section

LP and Its Dual

The standard form of LP is

$$\begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x}, \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

The dual problem is

$$\begin{array}{ll} \max & \boldsymbol{b}^T\boldsymbol{y}, \\ \text{subject to} & A^T\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c},\ \boldsymbol{s} \geq \boldsymbol{0}. \end{array}$$

Assumptions

Assume that

1. $\mathbf{rank}(A) = m$,
2. the primal problem has an optimal solution,
3. an initial BFS x^0 is available.

Let x^* be an optimal BFS of the primal problem, (y^*, s^*) be an optimal solution of the dual problem, and z^* be the optimal value.

Some Notations

Given a set of indices $B \subset \{1, 2, \ldots, n\}$, we split A, c, and x according to B and $N = \{1, 2, \ldots, n\} - B$ like

$$A = [A_B, A_N],\ c = \begin{bmatrix} c_B \\ c_N \end{bmatrix},\ x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}.$$

Then the standard form of LP is written as

$$\begin{array}{ll} \textbf{min} & c_B^T x_B + c_N^T x_N, \\ \textbf{subject to} & A_B x_B + A_N x_n = b, \\ & x_B \geq 0,\ x_N \geq 0. \end{array}$$

Basic Feasible Solutions (BFSs)

Define the set of bases

$$\mathcal{B} = \{B \subset \{1, 2, \ldots, n\} \mid |B| = m, \ \det(A_B) \neq 0\}.$$

Then the primal problem for the basis $B \in \mathcal{B}$ and $N = \{1, 2, \ldots, n\} - B$ can be written as

$$\begin{array}{ll} \min & c_B^T A_B^{-1} b + (c_N - A_N^T (A_B^{-1})^T c_B)^T x_N, \\ \text{subject to} & x_B = A_B^{-1} b - A_B^{-1} A_N x_N, \\ & x_B \geq 0, \ x_N \geq 0 \end{array}$$

which is called a dictionary.

Dictionary

Let $\bar{c}_{N^t} = c_{N^t} - A_{N^t}^T (A_{B^t}^{-1})^T c_{B^t}$ be the reduced cost vector, then the dictionary can be written as

$$\begin{array}{ll} \min & c_B^T A_B^{-1} b + \bar{c}_N^T x_N, \\ \text{subject to} & x_B = A_B^{-1} b - A_B^{-1} A_N x_N \geq 0, \\ & x_N \geq 0. \end{array}$$

The primal BFS is

$$x_B = A_B^{-1} b \geq 0, \; x_N = 0.$$

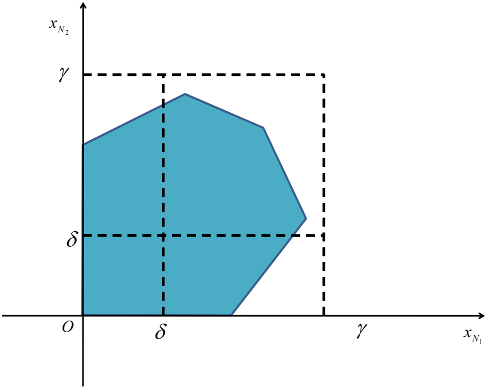
δ and γ

Let δ and γ be the minimum and the maximum values of all the positive elements of BFSs. Then for any BFS $\hat{\boldsymbol{x}}$ and any $j \in \{1, 2, \ldots, n\}$, we have

$$\delta \leq \hat{x}_j \leq \gamma \text{ if } \hat{x}_j \neq 0.$$

The values of δ and γ depend only on $\boldsymbol{A}$ and $\boldsymbol{b}$, but not on $\boldsymbol{c}$.

figure of δ, γ, and BFSs (vertices)



Pivoting

When $\bar{\boldsymbol{c}}_{N^t} \geq \boldsymbol{0}$, the current solution is optimal. Otherwise we conduct a pivot. Under Dantzig's rule, we choose a nonbasic variable whose reduced cost is minimum, i.e., we choose an index

$$j^t = \arg\min_{j \in N_t} \bar{c}_j.$$

Set $\Delta^t = -\bar{c}_{j^t} > 0$, that is, $-\Delta^t$ is the minimum value of the reduced costs.

Notations

$\boldsymbol{x}^*$	:	an optimal BFS of the primal
$(\boldsymbol{y}^*, \boldsymbol{s}^*)$	:	an optimal solution of the dual
$\boldsymbol{z}^*$	:	the optimal value
$\boldsymbol{x}^t$	:	the $\boldsymbol{t}$-th iterate of the simplex method
$\boldsymbol{B}^t$	:	the basis of $\boldsymbol{x}^t$
$\boldsymbol{N}^t$	:	the nonbasis of $\boldsymbol{x}^t$
$\bar{\boldsymbol{c}}_{N^t}$	:	the reduced cost vector at $\boldsymbol{t}$-th iteration
$\boldsymbol{\Delta}^t$	:	$-\min_{j \in N_t} \bar{c}_j$
$\boldsymbol{j}^t$	:	an index chosen by Dantzig's rule

New Section

A lower bound of z^*

We get a lower bound of the optimal value at each iteration of the simplex method.

Lemma

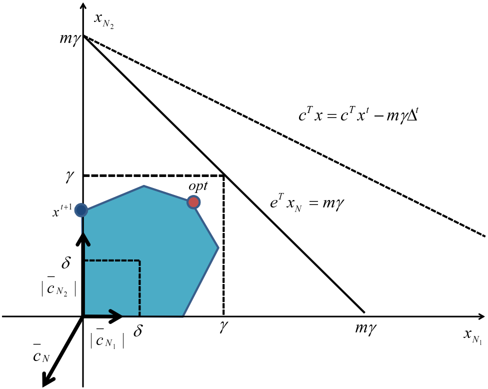
Let $\mathbf{x}^t$ *be the* t*-th iterate generated by the simplex method with Dantzig's rule. Then we have*

$$\boldsymbol{z^* \geq c^T x^t - \Delta^t m \gamma}$$

or the gap $(\mathbf{c}^T\mathbf{x}^t - z^*)$ *is bounded as follows*

$$\boldsymbol{\Delta^t m \gamma \geq c^T x^t - z^*}.$$

$$\begin{array}{ll} \min & c_{B^t}^T A_{B^t}^{-1} b + \bar{c}_{N^t}^T x_{N^t}, \\ \text{subject to} & A_{B^t}^{-1} b - A_{B^t}^{-1} A_{N^t} x_{N^t} \geq 0,\ x_{N^t} \geq 0. \end{array}$$



$$z^* \geq c^T x^t - \Delta^t m\gamma.$$

Proof: Let x^* be a basic optimal solution of the primal problem. Then we have

$$\begin{aligned} z^* &= c^T x^* \\ &= c_{B^t}^T A_{B^t}^{-1} b + \bar{c}_{N^t}^T x_{N^t}^* \\ &\geq c^T x^t - \Delta^t e^T x_{N^t}^* \\ &\geq c^T x^t - \Delta^t m \gamma, \end{aligned}$$

where the second inequality follows since x^* has at most m positive elements and each element is bounded above by γ. □

Reduction Rate of the Gap

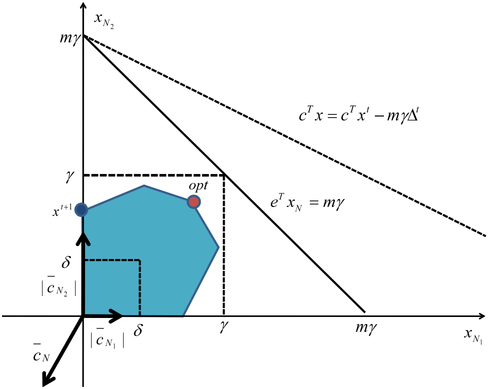
We show a constant reduction of the gap $(c^T x - z^*)$ when an iterate is updated. The reduction rate is independent of c.

Theorem

Let x^t and x^{t+1} be the t-th and $(t+1)$-th iterates generated by the simplex method with Dantzig's rule. If $x^{t+1} \neq x^t$, then we have

$$c^T x^{t+1} - z^* \leq \left(1 - \frac{\delta}{m\gamma}\right)(c^T x^t - z^*).$$

$$\begin{array}{ll} \min & c_{B^t}^T A_{B^t}^{-1} b + \bar{c}_{N^t}^T x_{N^t}, \\ \text{subject to} & A_{B^t}^{-1} b - A_{B^t}^{-1} A_{N^t} x_{N^t} \geq 0, \; x_{N^t} \geq 0. \end{array}$$



$$c^T x^{t+1} - z^* \leq \left(1 - \frac{\delta}{m\gamma}\right)(c^T x^t - z^*).$$

Proof. Let $x^t_{j^t}$ be the entering variable chosen at the t-th iteration. If $x^{t+1}_{j^t} = 0$, then we have $x^{t+1} = x^t$, a contradiction occurs. Thus $x^{t+1}_{j^t} \neq 0$, and we have $x^{t+1}_{j^t} \geq \delta$. Then we have

$$\begin{aligned} c^T x^t - c^T x^{t+1} &= \Delta^t x^{t+1}_{j^t} \\ &\geq \Delta^t \delta \\ &\geq \frac{\delta}{m\gamma}(c^T x^t - z^*). \end{aligned}$$

The desired inequality readily follows from the above inequality. □

Best Improvement Pivoting Rule

Under the best improvement pivoting rule, the objective function reduces at least as much as that with Dantzig's rule. So the next corollary follows.

Corollary

Let $\mathbf{x}^t$ and $\mathbf{x}^{t+1}$ be the t-th and $(t+1)$-th iterates generated by the simplex method with the best improvement rule. If $\mathbf{x}^{t+1} \neq \mathbf{x}^t$, then we also have

$$\mathbf{c}^T\mathbf{x}^{t+1} - z^* \leq \left(1 - \frac{\delta}{m\gamma}\right)(\mathbf{c}^T\mathbf{x}^t - z^*).$$

Number of solutions

From the constant reduction of the gap $\boldsymbol{c}^T\boldsymbol{x} - z^*$, we can get the following result.

Corollary

Let $\bar{\boldsymbol{x}}$ be a second optimal BFS of the primal problem. Then the number of different BFSs generated by the simplex method with Dantzig's rule (or the best improvement rule) is bounded by

$$\lceil m\frac{\gamma}{\delta} \log \frac{\boldsymbol{c}^T\boldsymbol{x}^0 - z^*}{\boldsymbol{c}^T\bar{\boldsymbol{x}} - z^*}\rceil.$$

An Upper Bound for an Element of BFS

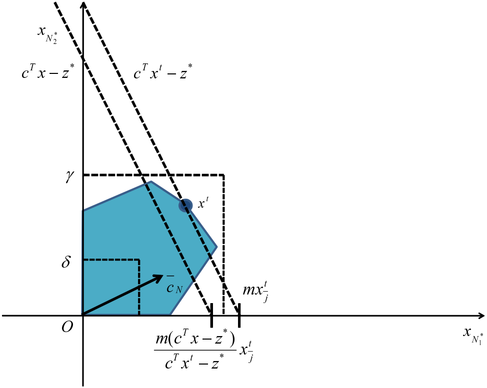
Lemma

Let $\mathbf{x}^t$ be the $\mathbf{t}$-th iterate generated by the simplex method. If $\mathbf{x}^t$ is not optimal, there exists $\bar{j} \in \mathbf{B}^t$ such that $\mathbf{x}^t_{\bar{j}} > \mathbf{0}$ and

$$\mathbf{x}_{\bar{j}} \leq \frac{\mathbf{c}^T\mathbf{x} - \mathbf{z}^*}{\mathbf{c}^T\mathbf{x}^t - \mathbf{z}^*} \mathbf{m} \mathbf{x}^t_{\bar{j}}$$

for any feasible solution $\mathbf{x}$.

$$\begin{array}{ll} \min & c_{B^*}^T A_{B^*}^{-1} b + \bar{c}_{N^*}^T x_{N^*}, \\ \text{subject to} & A_{B^*}^{-1} b - A_{B^*}^{-1} A_{N^*} x_{N^*} \geq 0,\ x_{N^*} \geq 0. \end{array}$$



$$x_{\bar{j}} \leq \frac{c^T x - z^*}{c^T x^t - z^*} m x_{\bar{j}}^t.$$

Proof. Since $c^T x^t - z^* = (x^t)^T s^* \leq m x^t_{\bar{j}} s^*_{\bar{j}}$ for some $\bar{j} \in B^t$, we have

$$s^*_{\bar{j}} \geq \frac{1}{m x^t_{\bar{j}}}(c^T x^t - z^*).$$

Since any feasible solution x satisfies $c^T x - z^* = x^T s^* \geq x_{\bar{j}} s^*_{\bar{j}}$, we have

$$x_{\bar{j}} \leq \frac{c^T x - z^*}{s^*_{\bar{j}}} \leq \frac{c^T x - z^*}{c^T x^t - z^*} m x^t_{\bar{j}}.$$

Element becomes Zero after Updates

Lemma

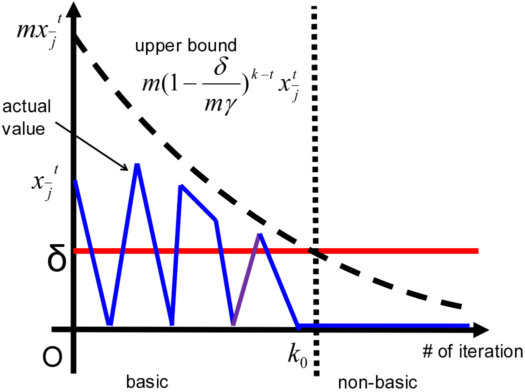
Let $\mathbf{x}^t$ be the t-th iterate when we generate a sequence of BFSs by the simplex method with Dantzig's rule. Assume that $\mathbf{x}^t$ is not an optimal solution. Then there exists $\bar{j} \in B^t$ satisfying

1. $x^t_{\bar{j}} > 0$.
2. *If the simplex method generates more than $\lceil m\frac{\gamma}{\delta} \log(m\frac{\gamma}{\delta}) \rceil$ different BFSs after t-th iterate, then $x_{\bar{j}}$ is zero.*

Proof. For $k \geq t+1$, let $\tilde{k}$ be the number of different BFSs appearing between the t-th and k-th iterations. Then there exists $\bar{j} \in B_t$ which satisfies

$$x_{\bar{j}}^k \leq m\left(1 - \frac{\delta}{m\gamma}\right)^{\tilde{k}} x_{\bar{j}}^t \leq m\gamma\left(1 - \frac{\delta}{m\gamma}\right)^{\tilde{k}}.$$

Therefore, if $\tilde{k} > m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})$, we have $x_{\bar{j}}^k < \delta$, which implies $x_{\bar{j}}^k = 0$ from the definition of δ. □



$x_{\bar{j}}^{k} = 0$ if the number of different BFSs appearing between the t-th and k-th iterations is more than $\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil$.

Bound for the Number of Solutions

Since the event described in Lemma can occur at most once for each variable, we get the next result.

Theorem

When we apply the simplex method with Dantzig's rule (or the best improvement rule) for LP having optimal solutions, for any $\boldsymbol{c}$ *the number of different BFSs is bounded by*

$$n\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil.$$

Note that the result is valid even if the simplex method fails to find an optimal solution because of a cycling.

Primal Nondegenerate Case

If the primal problem is nondegenerate, we have $\boldsymbol{x}^{t+1} \neq \boldsymbol{x}^{t}$ for all $\boldsymbol{t}$. This observation leads to a bound for the number of iterations of the simplex method.

Corollary

If the primal problem is nondegenerate, the simplex method finds an optimal solution in at most

$$n\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil \text{ iterations.}$$

New Section

The Dual Problem

$$\begin{array}{ll} \textbf{max} & b^T y, \\ \textbf{subject to} & A^T y + s = c, \ s \geq 0. \end{array}$$

Theorem

When we apply the (dual) simplex method with the most positive pivoting rule for the dual problem above, the number of different BFSs is bounded by

$$n \lceil m \frac{\gamma_D}{\delta_D} \log(m \frac{\gamma_D}{\delta_D}) \rceil,$$

where δ_D and γ_D are the minimum and maximum values of all the positive elements of dual BFSs.

LP with Bounded Variables

$$\begin{array}{ll} \min & c^T x, \\ \text{subject to} & Ax = b,\ 0 \le x \le u, \end{array}$$

Theorem

When we use the simplex method with Dantzig's rule for the LP above, the number of different BFSs is bounded by

$$n \lceil (n-m)\frac{\gamma}{\delta} \log((n-m)\frac{\gamma}{\delta}) \rceil.$$

In this case, $\gamma \le \max_i u_i$.

Primal Problem Again

Theorem

When we apply the simplex method with Dantzig's rule (or the best improvement rule) for the primal problem, for any $\mathbf{c}$ *the number of different BFSs is bounded by*

$$(n-m)\left\lceil \min\{m, n-m\}\frac{\gamma}{\delta}\log\left(\min\{m, n-m\}\frac{\gamma}{\delta}\right)\right\rceil.$$

Compare to the previous one

$$n\left\lceil m\frac{\gamma}{\delta}\log\left(m\frac{\gamma}{\delta}\right)\right\rceil.$$

New Section

0-1 Vertices

Assume that all the elements of BFSs (such as an assignment problem) is **0** or **1**, that is, $\delta = \gamma = 1$. Then the number of different BFSs generated by the simplex method with Dantzig's rule is bounded by

$$n \lceil m \log m \rceil.$$

Moreover, if $m = 1$, then the number is bounded by n. In this case, if we apply Theorem of reduction rate directly, then the number of updates is at most 1 (this bound is obviously tight).

Reduction Rate of the Gap

We show a constant reduction of the gap ($\boldsymbol{c}^T\boldsymbol{x} - z^*$) when an iterate is updated. The reduction rate is independent of $\boldsymbol{c}$.

Theorem

Let $\boldsymbol{x}^t$ and $\boldsymbol{x}^{t+1}$ be the t-th and $(t+1)$-th iterates generated by the simplex method with Dantzig's rule. If $\boldsymbol{x}^{t+1} \neq \boldsymbol{x}^t$, then we have

$$\boldsymbol{c}^T\boldsymbol{x}^{t+1} - z^* \leq \left(1 - \frac{\delta}{m\gamma}\right)(\boldsymbol{c}^T\boldsymbol{x}^t - z^*).$$

Shortest Path Problem

$$
\begin{array}{ll}
\min & \sum_{(i,j)\in E} c_{ij}x_{ij}, \\
\text{s.t.} & \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ij} = \begin{cases} |V|-1 & \text{for source} \\ -1 & \text{other nodes} \end{cases} \\
& x \geq 0.
\end{array}
$$

Since the shortest path problem is nondegenerate, $n = |E|$, $m = |V|$, $\gamma \leq |V| - 1$, and $\delta \geq 1$, the number of iterations of the simplex method with Dantzig's rule is bounded by

$$|E||V|^2 \log |V|^2.$$

Here we omit $\lceil \cdot \rceil$ for simplicity.

Minimum Cost Flow Problem

$$\begin{array}{ll} \min & \sum_{(i,j)\in E} c_{ij}x_{ij}, \\ \text{s.t.} & \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ij} = b_i \text{ for } i \in V \\ & 0 \le x \le u. \end{array}$$

Assume that the capacities u_{ij} and the supplies b_i are integral. Since $n = |E|$, $m = |V|$, $\gamma \le U = \max_{(i,j)\in E} u_{ij}$, and $\delta \ge 1$, the number of different solutions generated by the simplex method with Dantzig's rule is bounded by

$$(|E| - |V|)^2 U \log((|E| - |V|)U).$$

Minimum Cost Flow Problem (continue)

It is known that if we perturb the minimum cost flow problem by adding $-(|V|-1)/|V|$ to b_i for the root node and $1/|V|$ for the other nodes, then the problem is nondegenerate and we can solve the original problem by solving this perturbed problem. Hence the number of iterations of the simplex method with Dantzig's rule for solving a minimum cost flow problem is bounded by

$$(|E|-|V|)^2|V|U\log((|E|-|V|)|V|U).$$

Maximum Flow Problem

$$
\begin{array}{ll}
\max & f \\
\text{s. t.} & \sum_{\{j:(s,j)\in E\}} x_{sj} - \sum_{\{j:(j,s)\in E\}} x_{js} = f \\
& \sum_{\{j:(i,j)\in E\}} x_{ij} - \sum_{\{j:(j,i)\in E\}} x_{ji} = 0 \ \text{ for } i \neq s,t \\
& \sum_{\{j:(t,j)\in E\}} x_{tj} - \sum_{\{j:(j,t)\in E\}} x_{jt} = -f \\
& 0 \leq x_{ij} \leq u_{ij} \ \text{for } \forall (i,j) \in E
\end{array}
$$

Assume that all the capacities u_{ij} are integral. The dual is a minimum cut problem in which all the elements of BFSs are **0** or **1**. The number of different solutions generated by the dual simplex method is bounded by

$$(|E| - |V|)^2 \log(|E| - |V|).$$

A Totally Unimodular Matrix

We consider an LP whose constraint matrix $\boldsymbol{A}$ is totally unimodular and all the elements of $\boldsymbol{b}$ are integers. Then all the elements of any BFS are integers, so $\delta \geq 1$.
Let $(\boldsymbol{x}_B, \boldsymbol{0}) \in \Re^m \times \Re^{n-m}$ be a BFS of the LP. Then we have $\boldsymbol{x}_B = \boldsymbol{A}_B^{-1}\boldsymbol{b}$. Since $\boldsymbol{A}$ is totally unimodular, all the elements of $\boldsymbol{A}_B^{-1}$ are ± 1 or 0. Thus for any $j \in B$ we have $x_j \leq ||\boldsymbol{b}||_1$, which implies that $\gamma \leq ||\boldsymbol{b}||_1$.

A Bound in Totally Unimodular Case

Corollary

Assume that the constraint matrix $\boldsymbol{A}$ *is totally unimodular and the constraint vector* $\boldsymbol{b}$ *is integral. When we apply the simplex method with Dantzig's rule or the best improvement rule, we encounter at most*

$$nm\|\boldsymbol{b}\|_1 \log(m\|\boldsymbol{b}\|_1)$$

different BFSs.

MDP

The Markov Decision Problem (MDP):

$$\begin{array}{ll} \min & c_1^T x_1 + c_2^T x_2, \\ \text{subject to} & (I - \theta P_1) x_1 + (I - \theta P_2) x_2 = e, \\ & x_1, x_2 \geq 0, \end{array}$$

where P_1 and P_2 are $m \times m$ Markov matrices, θ is a discount rate, and e is the vector of all ones.

MDP has the following properties.

1. MDP is nondegenerate.
2. $\delta \geq 1$ and $\gamma \leq m/(1-\theta)$.

Number of Iterations for MDP

We obtain a similar result to Ye.

Corollary

The simplex method with Dantzig's rule for solving MDP finds an optimal solution in at most

$$n\frac{m^2}{1-\theta}\log\frac{m^2}{1-\theta}$$

iterations, where $n = 2m$.

New Section

Problems, Pivoting, and Assumptions

- Problems:
  1. The standard form of LP.
  2. Its dual.
- Pivoting rules:
  1. Dantzig's rule (the most negative reduced cost rule)
  2. The best improvement rule.
- Assumptions:
  1. $\mathrm{rank}(A) = m$.
  2. The primal problem has an optimal solution.
  3. An initial BFS is available.

Results

1. Constant reduction of the gap:

$$c^T x^{t+1} - z^* \leq \left(1 - \frac{\delta}{m\gamma}\right)(c^T x^t - z^*).$$

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2. The number of BFSs is bounded by

$$n\left\lceil m\frac{\gamma}{\delta}\log\left(m\frac{\gamma}{\delta}\right)\right\rceil.$$

Results

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$$n\left\lceil m\frac{\gamma}{\delta}\log\left(m\frac{\gamma}{\delta}\right)\right\rceil.$$

3. Totally unimodular case: It is bounded by

$$n\left\lceil m\|b\|_1 \log(m\|b\|_1)\right\rceil.$$

Results

1. Constant reduction of the gap:

$$\mathbf{c}^T\mathbf{x}^{t+1} - z^* \leq \left(1 - \frac{\delta}{m\gamma}\right)(\mathbf{c}^T\mathbf{x}^t - z^*).$$

2. The number of BFSs is bounded by

$$n\left\lceil m\frac{\gamma}{\delta}\log\left(m\frac{\gamma}{\delta}\right)\right\rceil.$$

3. Totally unimodular case: It is bounded by

$$n\left\lceil m\|\mathbf{b}\|_1 \log(m\|\mathbf{b}\|_1)\right\rceil.$$

4. MDP case: The number of iterations is bounded by

$$n\left\lceil \frac{m^2}{1-\theta}\log\frac{m^2}{1-\theta}\right\rceil.$$

Thank You for Your Attention!